Math 23 Review

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First Order Equations 1/3

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$f(x, y) = H(x)G(y) \implies$ Separation of variables

- Separate variables: move x's to one side, y's to the other

$$\frac{dy}{G(y)} = \frac{dx}{H(x)}$$

- Integrate both sides and solve for $y(x)$.

First Order Equations 2/3

Integrating factors for first order linear equations

- For an equation of the form:

$$y'(t) + p(t)y(t) = g(t)$$

- Form integrating factor:

$$\mu(t) = e^{\int p(t)\, dt}$$

- Solution:

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)g(t)\, dt$$

First Order Equations 3/3

Exact equations
Form:

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

- If $M_y = N_x$ then we can find solution $\psi(x, y) = 0$ via integration:
- $\psi_x = M$, $\psi_y = N$
- Integrate both and form the most general $\psi(x, y)$.

Second Order Equations: Linear homogeneous e

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$$p(x)y''(t) + q(x)y'(t) + r(x)y(t) = 0$$

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General facts:

- Superposition: If $y_1(t)$ and $y_2(t)$ are linearly independent solutions then the general solution is

$$C_1y_1(t) + C_2y_2(t)$$

- Initial value problem: If we are given two initial conditions and a fundamental set of solutions $\{y_1(t), y_2(t)\}$ then we can solve the initial value problem.

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$$ay'' + by' + cy = 0$$

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- Auxillary equation: $ar^2 + br + c = 0$
- Roots:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Classification according to roots

- If $b^2 - 4ac > 0$ then the solution is of the form

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- If $b^2 - 4ac < 0$ then $r = a \pm ib$ and the solution is of the form

$$y(t) = e^{at}(C_1 \cos(bt) + C_2 \sin(bt))$$

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- Solve this separable equation for $v(t)$

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- If $y_p(t)$ is a particular solution and $\{y_1(t), y_2(t)\}$ is a fundamental set of solutions for the homogeneous equation then the general solution is

$$y(t) = C_1 y_1(t) + C_2 y_2(t) + y_p(t)$$

Finding a particular solution

- Method of undetermined coefficients

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- Variation of parameter

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- See table on page 175.

Variation of Parameters

Given an equation

$$y'' + p(t)y' + q(t)y = g(t)$$

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- Assume $u_1' y_1 + u_2' y_2 = 0$
- Plugging into the equation and simplifying yields a differential equation:

$$u_1' y_1' + u_2' y_2' = g(t)$$

Variation of Parameters

Solving these two equations yields

$$y_p(t) = -y_1(t) \int \frac{y_2(t)g(t)}{Wr(y_1, y_2)(t)}\, dt + y_2(t) \int \frac{y_1(t)g(t)}{Wr(y_1, y_2)(t)}\, dt$$

Series solutions to second order equations

General equation:

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

If x_0 is an ordinary point of the equation, then we can find a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

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Method of solution:

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- Simplify and reindex, writing the equation as a single series
- Set each coefficient to zero to find relations between the $\{a_n\}$
- Using the recurrence relation, find the general series form of the solution.

First order linear systems

General Form:

$$\vec{x}' = A\vec{x}$$

where A is a matrix.

Solutions depend on eigenvectors and eigenvalues of the matrix A

Systems: Method of solution

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  Solve: $det(A - \lambda I) = 0$ for all values of λ

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- If there are no repeated eignevalues, the solution is then of the form:

$$\vec{x} = C_1\vec{\xi_1}e^{\lambda_1 t} + \cdots + C_n\vec{\xi_n}e^{\lambda_n t}$$

Repeated Eigenvalues

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  where λ is the repeated eigenvalue and $\vec{\xi}$ is an eigenvector for λ.
- Then, the piece of the general solution associated to λ is

$$\vec{x}(t) = \vec{\eta} t e^{\lambda t} + \vec{\xi} e^{\lambda t}$$

Qualitative analysis of solutions

If A is a 2x2 matrix then we can draw a phase portrait of the system and analyze the behavior of solutions qualitatively. The portraits can be sketched and classified using the eigenvalue and eigenvector data from A.

Classification of 2x2 systems

If the eigenvalues are λ_1, λ_2 with eigenvectors ξ_1, ξ_w, then

- Real distinct eigenvalues

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- $0 < \lambda_1 < \lambda_2$: the line along ξ_2 is a node, paths move away from the node - unstable.
- $0 > \lambda_1 > \lambda_2$: the line along ξ_2 is a node, paths move towards from the node - stable (figure 9.1.1 (a)).
- Opposite sign: saddle - unstable, see figure 9.1.2 (a)

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- $a > 0$ - unstable
- $a < 0$ - stable
- $a = 0$ - stable center

Repeated eigenvalue

$$\lambda_1 = \lambda_2$$

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- $\lambda_1 > 0$: proper or improper unstable node (e.g. figure 9.1.3 (a))
- $\lambda_1 < 0$: proper or improper stable node (e.g. figure 9.1.4 (a))

Almost linear systems

Systems of the form

$$\vec{x}' = A\vec{x} + g(\vec{x})$$

can be qualitatively classified in terms of the eignevalues/eignevectors of A. See table 9.3.1

Partial Differential Equations

Three fundamental equations:

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- The heat equation:

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- Laplace's equation:

$$u_{xx} + u_{yy} = 0$$

Separation of variables

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- Plug in and separate variables
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- Impose boundary conditions to form a two point boundary value problem

Two point boundary value problems and Fourier

For example:

$$X'' + \lambda X = 0$$

$$X(0) = 0, \;\; X(L) = 0$$

- Find eigenvalues and eigenfunctions

Two point boundary value problems and Fourier

For example:

$$X'' + \lambda X = 0$$

$$X(0) = 0, \;\; X(L) = 0$$

- Find eigenvalues and eigenfunctions
- Superimpose all solutions to form a Fourier series solution

Fourier Coefficients

- Calculate Fourier coefficients using integral formulae:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi}{L}x)\;dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi}{L}x)\;dx$$

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- Remeber tricks for Fourier $\sin$ and $\cos$ series