Math 23 Spring 2009 Final Exam

Instructor (circle one): Chernov, Sadykov

Friday June 5, 2009

3-6 PM Moore Hall, Filene auditorium

PRINT NAME: ______________________________

Instructions: This is a closed book, closed notes exam. **Use of calculators is not permitted. You must justify all of your answers to receive credit.**

You have **three hours**. Do all the problems. Please do all your work on the paper provided.

The Honor Principle requires that you neither give nor receive any aid on this exam.

Grader's use only:

1. ________ /10

2. ________ /10

3. ________ /10

4. ________ /10

5. ________ /10

6. ________ /10

7. ________ /10

8. ________ /10

9. ________ /10

10. ________ /10

Total: ________ /100

1. Solve the boundary value problem

$$y'' + 4y = 0, \qquad y'(0) = 1, \quad y(\pi) = 0.$$

2. (10 points) Consider the heat conduction in a rod 20 cm in length whose ends are maintained at zero temperature for all $t > 0$. Suppose that $\alpha^2 = 4$. Find an expression for the temperature $u(x,t)$ if the initial temperature distribution is $u(x,0) = f(x) = 2x$, for $0 < x < 20$. **Hint:** in such cases the solution $u(x,t)$ can be found as a series $\sum_{n=1}^{\infty} a_n \sin\left(\frac{\pi n x}{L}\right) e^{-\frac{\pi^2 n^2 a^2}{L^2} t}$

3. **(a, 3 points)** Find the recurrence equation for coefficients of the series solution of

$$y'' + xy' + 2y = 0, \text{ about the given point } \qquad x_0 = 0.$$

(b, 3 points) Find the first four terms in each of two solutions y_1, y_2 (unless the series terminates sooner).

(c, 4 points) Find the general term in each solution.

4. **(a, 6 points)** Determine the general solution of the given Euler differential equation

$$x^2y'' - 5xy' + 9y = 0$$

that is valid in any interval not including the singular point.

(b, 4 points) Solve the initial value problem $y(1) = 3$ and $y'(1) = 9$.

5. **(a, 5 points)** Find the general solution of the homogeneous equation $y^{(6)} + y^{(3)} = 0$.

(b, 2 points) Determine the suitable form for a particular solution $Y(t)$ in the method of undetermined coefficients in the equation $y^{(6)} + y^{(4)} = 5t^2$. Do not attempt to find $Y(t)$ explicitly.

(c, 1 points) Determine the suitable form for a particular solution $Y(t)$ in the method of undetermined coefficients in the equation $y^{(6)} + y^{(4)} = 32e^{\frac{1}{2}t}$. Do not attempt to find $Y(t)$ explicitly.

(d, 2 points) Determine the suitable form for a particular solution $Y(t)$ in the method of undetermined coefficients in the equation $y^{(6)} + y^{(4)} = 239e^{\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right)$. Do not attempt to find $Y(t)$ explicitly.

6. Determine the behavior of the function y as $t \to \infty$, where y is the solution of the initial value problem

$$y' = y(4 - y)(5 - y), \quad y(0) = 3.$$

(Hint: construct the direction field first.)

7. Determine if each of the equations is exact. If it is exact, find the solution.

a (5 points) $(e^x \sin y + 3y) - (3x - e^x \sin y)\frac{dy}{dx} = 0.$

b (5 points) $(e^x \sin y - 2y \sin x) + (e^x \cos y + 2\cos x)\frac{dy}{dx} = 0.$

8. Solve the initial value problem:

$$y'' - 2y' + y = te^t + 4, \qquad y(0) = 1, \quad y'(0) = 1.$$

9. Solve the initial value problem:

$$\mathbf{x}' = \begin{pmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 7 \\ 5 \\ 5 \end{pmatrix}.$$

10. (10 points) Consider an elastic string of length 2π and the constant $a = 3$. The string is set in motion from its equilibrium position with an initial velocity $u_t(x, 0) = g(x)$, where

$$g(x) = \begin{cases} x & \text{if } 0 \leq x \leq \pi \\ 2\pi - x & \text{if } \pi \leq x \leq 2\pi \end{cases}$$

Find the displacement function $u(x, t)$. **Hint:** in such cases the solution $u(x, t)$ can be found as a series $\sum_{n=1}^{\infty} d_n \sin\left(\frac{\pi n x}{L}\right) \sin\left(\frac{\pi n a t}{L}\right)$