Math 23: Linear Algebra
In-Class Problem
Due Wednesday, May 29

This problem refers to the Hom vector spaces of the last long homework. We're going to look in detail at $\mathrm{Hom}(V, \mathbb{R})$. This vector space occurs very often in lots of situations, including geometry and physics. It is called the **dual vector space** to V, and usually denoted V^*. In other words,

$$V^* = \mathrm{Hom}(V, \mathbb{R}).$$

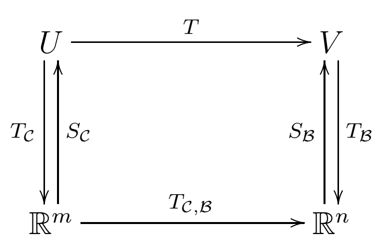
We'll want to get a basis for V^*. For this, we need a basis for $\mathbb{R}$ and a basis for V. We'll use the standard basis $\{\mathbf{e}_1\}$ for $\mathbb{R}$, and let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for V. You showed in Long Homework 2 that the collection $\{T_{11}, T_{21}, \ldots, T_{n1}\}$ is a basis for V^*. For simplicity, these linear transformations are usually denoted $\mathcal{B}^* = \{\mathbf{b}_1^*, \mathbf{b}_2^*, \ldots, \mathbf{b}_n^*\}$

Here are some questions just to get your mind working, what is $\mathbf{b}_i^*(\mathbf{b}_j)$? If V is 5-dimensional, then what is the dimension of V^*? Are V and V^* isomorphic?

You also showed in Long Homework 2 that if you have a linear transformation $T : U \longrightarrow V$, then it induces a linear transformation

$$T^* : V^* \longrightarrow U^* \quad \text{given by the formula} \quad T^*(S) = S \circ T.$$

Let's give U a basis $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_m\}$ (so $\mathcal{C}^* = \{\mathbf{c}_1^*, \mathbf{c}_2^*, \ldots, \mathbf{c}_m^*\}$ is a basis for for U^*). Then we can find a matrix for T with respect to the bases $\mathcal{B}$ and $\mathcal{C}$, in the usual way. More explicitly, we construct a diagram



and we can get a matrix A for the linear transformation $T_{\mathcal{B},\mathcal{C}}$.

Your Task Determine the matrix, relative to the bases $\mathcal{B}^*$ and $\mathcal{C}^*$ for the linear transformation $T^* : V^* \longrightarrow U^*$. For simplicity, you may restrict your attention to linear transformations $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, using the standard bases.