Math 23: Linear Algebra

Long Homework Assignment 1

Due Friday, April 27

Determinants

The determinant is a function that takes $n \times n$ matrices and gives you numbers. (If you want to, you can think of it as a function $\text{Det} : \mathbb{R}^{n^2} \longrightarrow \mathbb{R}$.) In this assignment you'll prove that the determinant is *characterized* by three simple properties. This means that if you have in your hand a function, and you can check that it satisfies those properties, then you know that it is none other than the determinant.

To write them down, we need to think of our matrix in terms of rows instead of columns. Thus

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{12} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} (a_{11}, a_{12}, \cdots, a_{1n}) \\ (a_{21}, a_{12}, \cdots, a_{2n}) \\ \vdots \\ (a_{n1}, a_{n2}, \cdots, a_{nn}) \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_n \end{pmatrix}$$

so $\mathbf{a}_i$ is the i^{th} row of the matrix A.

Now we have to define the determinant! Let D be a function which assigns a number to each $n \times n$ matrix. Then D is the determinant if the following three conditions hold.

1. D is **multilinear**. This means that if you think one row at a time, then D is a linear transformation. More explicitly, D should

   (a) respect addition in a single row:

$$D \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_i + \mathbf{b}_i \\ \vdots \\ \mathbf{a}_n \end{pmatrix} = D \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_i \\ \vdots \\ \mathbf{a}_n \end{pmatrix} + D \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{b}_i \\ \vdots \\ \mathbf{a}_n \end{pmatrix}$$

(b) respect multiplication by a constant in a row:

$$D \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ c\mathbf{a}_i \\ \vdots \\ \mathbf{a}_n \end{pmatrix} = c\, D \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_i \\ \vdots \\ \mathbf{a}_n \end{pmatrix}$$

2. D is **alternating**. More precisely, if you interchange two rows, the determinant changes sign:

$$D \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_i \\ \vdots \\ \mathbf{a}_j \\ \vdots \\ \mathbf{a}_n \end{pmatrix} = -D \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_j \\ \vdots \\ \mathbf{a}_i \\ \vdots \\ \mathbf{a}_n \end{pmatrix}$$

3. All of this defines the determinant, up to a scalar multiple. To be sure that we have the determinant, we need to check that $D(I_n) = 1$.

The first thing to do is to check that this definition *makes sense*. It is entirely possible (at least imaginable) that there are millions and millions of different functions which all have these four properties. We have to show that, in fact, there is only one function D which has all four of those properties.

Once this is done, you will be able to tell if a given function on matrices is the determinant: simply check to see if conditions 1 through 3 hold.

1 Alternating Multilinear Functions

To begin with, we will forget about condition 3 and concentrate on functions which are alternating and multilinear. For all of this section, let D be a fixed alternating and multilinear function.

1.1 Elementary Row Operations

In this first part, you will figure out how the an alternating multilinear function changes when you perform the various elementary row operations. Some of them are automatic: if you interchange two row, then the sign changes, by condition 2; if you multiply a row by a (nonzero) constant c, then the determinant is multiplied by c by 1(b). So it remains to find out what happens when you add a multiple of one row to another.

Problem 1 Show that if an $n \times n$ matrix A has two identical rows, then $D(A) = 0$.

Solution For this we need to use the fact that D is alternating. We can write A in the form

$$\begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_i = \mathbf{b} \\ \vdots \\ \mathbf{a}_j = \mathbf{b} \\ \vdots \\ \mathbf{a}_n \end{pmatrix}$$

(in other words, $\mathbf{a}_i$ and $\mathbf{a}_j$ are the same vector, which I'm calling $\mathbf{b}$). Therefore

$$D(A) = D\begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_i = \mathbf{b} \\ \vdots \\ \mathbf{a}_j = \mathbf{b} \\ \vdots \\ \mathbf{a}_n \end{pmatrix} = -D\begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_j = \mathbf{b} \\ \vdots \\ \mathbf{a}_i = \mathbf{b} \\ \vdots \\ \mathbf{a}_n \end{pmatrix} = -D(A)$$

because the matrix A is *unchanged* when the i^{th} row and the j^{th} row are interchanged. Since $D(A) = -D(A)$, the only possibility is that $D(A) = 0$.

Problem 2 Suppose the matrix B is obtained from the matrix A by adding a multiple of one row to another. Show that $D(B) = D(A)$.

Solution We can calculate using linearity and Problem 1:

$$D\begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_i + c\mathbf{a}_j \\ \vdots \\ \mathbf{a}_j \\ \vdots \\ \mathbf{a}_n \end{pmatrix} = D\begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_i \\ \vdots \\ \mathbf{a}_j \\ \vdots \\ \mathbf{a}_n \end{pmatrix} + cD\begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_j \\ \vdots \\ \mathbf{a}_j \\ \vdots \\ \mathbf{a}_n \end{pmatrix} = D(A) + 0 = D(A).$$

Now you know what each elementary row opertation does to the determinant of a matrix. Fill in the following table for easy reference.

Operation	**Effect on D**
Replacement	**D(B) = D(A)**
Scaling by c	**D(B) = c D(A)**
Interchange	**D(B) = -D(A)**

(Here B is the result of doing a single operation to A.)

1.2 Upper Triangular Matrices

The *diagonal* of a matrix A is the list of entries $a_{11}, a_{22}, \ldots, a_{nn}$. A matrix A is a *diagonal matrix* if all of the nondiagonal entries are zero. Of particular importance is the *identity matrix*

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Problem 3 If A is the diagonal matrix

$$A = \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

show that $D(A) = a_{11}a_{22}\cdots a_{nn}D(I_n)$.

Solution For this we use the linearlity property repeatedly, as follows

$$D\begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} = a_{11}D\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

$$= a_{11}a_{22}D\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} = \cdots = a_{11}a_{22}\cdots a_{nn}D\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

(If you wanted to be more formal, you could simplify this via an inductive proof.)

Notice that this proves that the determinant of a diagonal matrix is simply the product of the diagonal entries. Next, you need to prove the same is true for upper triangular matrices.

A matrix A is *upper triangular* if all of the entries *below* the diagonal are zero. For example, any $n \times n$ matrix in echelon form is an upper triangular matrix.

Problem 4 If A is the upper triangular matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{2n} \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

show that $D(A) = a_{11}a_{22}\cdots a_{nn}D(I_n)$.

Solution First we'll show that if A has any zeros on the diagonal, then $D(A) = 0$. In fact, if A has any zeros on the diagonal, then A must have fewer than n pivot positions, so A is row equivalent to a matrix with a row of zeros. By the multilinearity of D, $D(B) = 0$ if B is a matrix with a row of zeros (pull out a factor of zero from the zero row). Since the effect of elementary row operations on D is to multiply by nonzero constants, any matrix A that is row equivalent to a matrix with a row of zeros must also have $D(A) = 0$. Now if all the a_{ii} are nonzero, then the matrix can be reduced to a diagonal matrix by replacements, which do not change the value of D. Thus

$$\begin{aligned} D\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{2n} \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} &= D\begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} \\ &= a_{11}a_{22}\cdots a_{nn}D(I_n) \end{aligned}$$

by Problem 3.

1.3 Putting It All Together

Now you are ready to prove the main theorem about determinants.

Theorem 1 Let D be an alternating multilinear function.

(a) Then D is entirely determined by $D(I_n)$.

(b) The determinant det is in fact entirely determined by the properties 1 through 3, and $D(A) = \det(A)D(I_n)$ for every matrix A.

The hard part of the proof is part (a). This part will be proved by induction on the number, k, of elementary row operations it takes to reduce A echelon form. A proof by induction involves two steps, as outlined below.

Initial Step Prove that theorem is true if $k = 0$. In other words, prove that, if A is *already* in echelon form, then $D(A)$ is determined by $D(I_n)$. You have already done this, in Problem 4.

Inductive Step Assume now that the theorem is known for any matrix A that can be reduced to echelon form in k or fewer steps. Now prove that the theorem is true for any matrix A that can be reduced to echelon form in $k + 1$ steps.

Problem 5 Prove the inductive step. (HINT What is the first step in the reduction? After you have done that first step, how many steps are left to do?)

Solution We are assuming now that we have two multilinear alternating functions, D and $\widetilde{D}$, and that $D(I_n) = \widetilde{D}(I_n)$. We are also assuming that, if B can be reduced to echelon form in k or fewer steps, then $D(B) = \widetilde{D}(B)$ (this is the *inductive hypothesis*). Now we let A be some matrix that requires $k + 1$ steps to reduce to echelon form. We have to show that $D(A) = \widetilde{D}(A)$. Let B be the result of the first step in the reduction of A to echelon form. Then B only requires k steps to reduce to echelon form, and so $D(B) = \widetilde{D}(B)$ by the inductive hypothesis. Notice that we never have to use a scaling operation to reduce a matrix to (nonreduced) echelon form, so we really only have two possiblities for the first operation.

1. Suppose the first operation used to reduce A to echelon form is a replacement operation. Then we have

$$D(A) = D(B) = \widetilde{D}(B) = \widetilde{D}(A)$$

   since replacement has no effect on the value of D or $\widetilde{D}$.

2. Suppose the first operation used to reduce A to echelon form is an interchange operation. Then we have

$$D(A) = -D(B) = -\widetilde{D}(B) = \widetilde{D}(A)$$

   since interchange reverses the sign of both D and $\widetilde{D}$.

In either case, we have $D(A) = \widetilde{D}(A)$.

Problem 6 Explain in detail why the work you have done so far proves the theorem.

Solution First we'll explore part (a). Let A be any matrix. Then A can be reduced to echelon form in some number of steps, call it k. If $k = 0$ (i.e., A is *already in* echelon form), then Problem 4 shows that $D(A) = \widetilde{D}(A)$. In other words, $D(A) = \widetilde{D}(A)$ for any matrix A which takes at most 0 steps to reduce to echelon form. By Problem 5, it follows that $D(A) = \widetilde{D}(A)$ for any matrix A which takes at most 1 step to reduce to echelon form. By Problem 5 again, (with $k = 1$ this time) $D(A) = \widetilde{D}(A)$ for any matrix A which takes at most 1 step to reduce to echelon form. Continuing in this way, we see that $D(A) = \widetilde{D}(A)$ for any matrix A. Now let's think about (b). Since det is multilinear and alternating by definition, it will be completely determined once we say what $\det(I_n)$ is. Property 3 tells us $\det(I_n) = 1$, so det is completely determined by properties 1 through 3. To prove the last formula, let $\widetilde{D}(A) = \det(A)D(I_n)$. Then

$$\widetilde{D}(I_n) = \det(I_n)D(I_n) = D(I_n),$$

so we can conclude (by part (a) of Theorem 1) that $D(A) = \widetilde{D}(A) = \det(A)D(I_n)$ as soon as we show that $\widetilde{D}$ is alternating and multilinear. Since we'll use this fact later, I'll separate it out for easy reference.

Lemma If c is any constant then the function $\widetilde{D}(A) = c \cdot \det(A)$ is alternating and multilinear.

Proof We just have to check the properties. First of all,

$$\widetilde{D}\begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ s\mathbf{a}_i + t\mathbf{b}_i \\ \vdots \\ \mathbf{a}_n \end{pmatrix} = c\det\begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ s\mathbf{a}_i + t\mathbf{b}_i \\ \vdots \\ \mathbf{a}_n \end{pmatrix} = c\left(s\det\begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_i \\ \vdots \\ \mathbf{a}_n \end{pmatrix} + t\det\begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{b}_i \\ \vdots \\ \mathbf{a}_n \end{pmatrix}\right)$$

$$= sc\det\begin{pmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \vdots\\ \mathbf{a}_i\\ \vdots\\ \mathbf{a}_n\end{pmatrix} + tc\det\begin{pmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \vdots\\ \mathbf{b}_i\\ \vdots\\ \mathbf{a}_n\end{pmatrix} = s\widetilde{D}\begin{pmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \vdots\\ \mathbf{a}_i\\ \vdots\\ \mathbf{a}_n\end{pmatrix} + t\widetilde{D}\begin{pmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \vdots\\ \mathbf{b}_i\\ \vdots\\ \mathbf{a}_n\end{pmatrix},$$

So $\widetilde{D}$ is multilinear. Also,

$$\widetilde{D}\begin{pmatrix}\mathbf{a}_1\\ \vdots\\ \mathbf{a}_i\\ \vdots\\ \mathbf{a}_j\\ \vdots\\ \mathbf{a}_n\end{pmatrix} = c\det\begin{pmatrix}\mathbf{a}_1\\ \vdots\\ \mathbf{a}_i\\ \vdots\\ \mathbf{a}_j\\ \vdots\\ \mathbf{a}_n\end{pmatrix} = -c\det\begin{pmatrix}\mathbf{a}_1\\ \vdots\\ \mathbf{a}_j\\ \vdots\\ \mathbf{a}_i\\ \vdots\\ \mathbf{a}_n\end{pmatrix} = -\widetilde{D}\begin{pmatrix}\mathbf{a}_1\\ \vdots\\ \mathbf{a}_j\\ \vdots\\ \mathbf{a}_i\\ \vdots\\ \mathbf{a}_n\end{pmatrix}$$

which proves the Lemma.

Since the determinant of A ultimately depends on an echelon form of A, we can think about it in terms of pivot positions.

Problem 7 Show that $\det(A) \neq 0$ if and only if the columns of A are linearly independent.

Solution If $\det(A) \neq 0$, then any matrix B that is row equivalent to A also has $\det(B) \neq 0$, because according the table you filled in, the effect of any row operation on det is to multiply by a *nonzero* constant. So it must be true that the reduced echelon form of A has nonzero determinant. Since an echelon form of A is an upper triangular matrix, you can see by Problem 3 that the reduced echelon form of A has no zeros on the diagonal – in other words, A is row equivalent to the identity matrix. On the other hand, if A is row equivalent to the identity, then A can be obtained from I_n by row operations, and so $\det(A) \neq 0$ since $\det(I_n) \neq 0$. Summarizing: $\det(A) \neq 0$ if and only if A is row equivalent to the identity. On the third hand, we know that A is row equivalent to the identity if and only if the columns of A are linearly independent, since there is a pivot position in every column.

2 Computing Determinants

As a byproduct of what you have done, you have discovered a way to compute the determinant of any matrix! Simply reduce it to echelon form, keeping track of what each step does to the determinant.

Problem 8 Compute the determinant of the matrix

$$A = \begin{pmatrix} 1 & 0 & 3 & 4 \\ 2 & 1 & 5 & 1 \\ 3 & 2 & -1 & -1 \\ -1 & 0 & 4 & 1 \end{pmatrix}$$

Show your work.

Solution It turns out that $\det(A) = -47$.

3 Determinants and Matrix Multiplication

One of the most important properties of the determinant function is the fact that it respects matrix multiplication.

Theorem 2 For any two $n \times n$ matrices A and B, $\det(A \cdot B) = \det(A)\det(B)$.

Problem 9 Prove it! (HINT Define $D(A) = \det(A \cdot B)$. Check that D is alternating and bilinear and apply Theorem 1. What is $D(I_n)$?)

Solution In this problem we'll think of A as a variable matrix and B as a constant. Then the function $D(A) = \det(A \cdot B)$ is a perfectly good function, as is the fucntion $\widetilde{D}(A) = \det(A)\det(B)$. We'll show that these two functions are the same, and that will prove that $\det(A \cdot B) = \det(A)\det(B)$ no matter what A is. And since B can be chosen (in advance) to be any matrix, this will prove Theorem 2. To show these functions are the same, we'll use Theorem 1. We'll simply show that D and $\widetilde{D}$ are both multilinear and alternating, and that $D(I_n) = \widetilde{D}(I_n)$. Theorem 1 then guarantees that $D(A) = \widetilde{D}(A)$ for every matrix A. Since $\widetilde{D}$ is a constant multiple of the determinant, it is multilinear and alternating (see Problem 6). So we just have to show that D is multilinear and alternating. Let's go:

$$D\begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ s\mathbf{a}_i + t\mathbf{b}_i \\ \vdots \\ \mathbf{a}_n \end{pmatrix} = \det\left(\begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ s\mathbf{a}_i + t\mathbf{b}_i \\ \vdots \\ \mathbf{a}_n \end{pmatrix} B\right) = \det\begin{pmatrix} \mathbf{a}_1 B \\ \mathbf{a}_2 B \\ \vdots \\ (s\mathbf{a}_i + t\mathbf{b}_i)B \\ \vdots \\ \mathbf{a}_n B \end{pmatrix}$$

$$= \det\begin{pmatrix} \mathbf{a}_1 B \\ \mathbf{a}_2 B \\ \vdots \\ s\mathbf{a}_i B + t\mathbf{b}_i B \\ \vdots \\ \mathbf{a}_n B \end{pmatrix} = \det s\begin{pmatrix} \mathbf{a}_1 B \\ \mathbf{a}_2 B \\ \vdots \\ \mathbf{a}_i B \\ \vdots \\ \mathbf{a}_n B \end{pmatrix} + t\det\begin{pmatrix} \mathbf{a}_1 B \\ \mathbf{a}_2 B \\ \vdots \\ \mathbf{b}_i B \\ \vdots \\ \mathbf{a}_n B \end{pmatrix} = sD\begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_i \\ \vdots \\ \mathbf{a}_n \end{pmatrix} + tD\begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{b}_i \\ \vdots \\ \mathbf{a}_n \end{pmatrix}.$$

This proves that D is multilinear. To show that D is alternating, we do a similar calculation

$$D\begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_i \\ \vdots \\ \mathbf{a}_j \\ \vdots \\ \mathbf{a}_n \end{pmatrix} = \det\begin{pmatrix} \mathbf{a}_1 B \\ \vdots \\ \mathbf{a}_i B \\ \vdots \\ \mathbf{a}_j B \\ \vdots \\ \mathbf{a}_n B \end{pmatrix} = -\det\begin{pmatrix} \mathbf{a}_1 B \\ \vdots \\ \mathbf{a}_j B \\ \vdots \\ \mathbf{a}_i B \\ \vdots \\ \mathbf{a}_n B \end{pmatrix} = -D\begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_j \\ \vdots \\ \mathbf{a}_i \\ \vdots \\ \mathbf{a}_n \end{pmatrix}.$$

The last thing to do is to notice that

$$D(I_n) = \det(I_n B) = \det(B) = \det(I_n)\det(B) = \widetilde{D}(I_n),$$

so $D(A) = \widetilde{D}(A)$ for every matrix A by Theorem 1.

Finally, let's derive a useful consequence of Theorem 2.

Problem 10 Show that if A is invertible, then $\det(A) \neq 0$. Give (and prove) a formula for $\det(A^{-1})$ in terms of $\det(A)$.

Solution Since A is invertible, there is a matrix A^{-1} such that $AA^{-1} = I_n$. By Theorem 2, we have

$$1 = \det(I_n) \det(AA^{-1}) = \det(A) \det(A^{-1}),$$

so

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Obviously, this number cannot be zero.