# Vector spaces WORKSHEET

10/21/03

A **vector space** is a nonempty set $V$ of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below.[1] The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $V$ and for all scalars $c$ and $d$.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in $V$.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$. (addition is commutative)
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$. (addition is associative)
4. There is a **zero** vector $\mathbf{0}$ in $V$ such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each $\mathbf{u}$ in $V$, there is a vector $-\mathbf{u}$ in $V$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. The scalar multiple of $\mathbf{u}$ by $c$, denoted by $c\mathbf{u}$, is in $V$.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$. (scalar multiplication is associative)
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$. "
9. $c(d\mathbf{u})=(cd)\mathbf{u}$. "
10. $1\mathbf{u}=\mathbf{u}$.

Note: so all elements of H have to be in V.

A **subspace** of a vector space $V$ is a $\boxed{\text{subset}}$ $H$ of $V$ that has three properties:

a. The zero vector of $V$ is in $H$.[2]

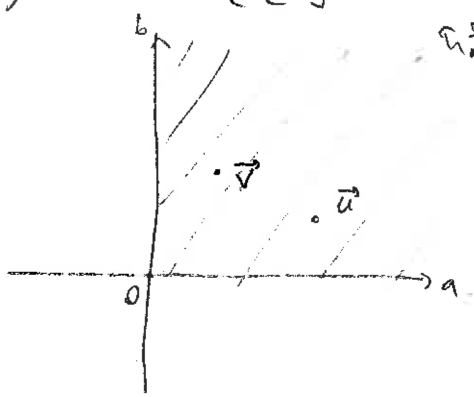
b. $H$ is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in $H$, the sum $\mathbf{u}+\mathbf{v}$ is in $H$.

c. $H$ is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in $H$ and each scalar $c$, the vector $c\mathbf{u}$ is in $H$.

Let's take $V = \mathbb{R}^2$, and do the above tests on some candidate subspaces $H$, to see if they really <u>are</u> valid subspaces:

A) $H = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : a \geq 0,\ b \geq 0 \right\}$

simply means, all points in the first quadrant




Is it a subset? (T/F)
Property a) (T/F)
b) (T/F)
c) (T/F)

so, is H a subspace? (Y/N)

Hint: properties must be true for <u>all</u> scalars $c$, vectors $\vec{u}, \vec{v}$.

B) $H$ = all vectors of the form $\begin{bmatrix} 1+a \\ 2a \end{bmatrix}$ where $a$ is a free parameter.

subset T/F a) T/F , b) T/F , c) T/F

C) $H$ = all vectors of the form $\begin{bmatrix} 1+a \\ 2+2a \end{bmatrix}$

D) $H = \mathbb{R}^1$ ie one-component vectors

[Hint: is it a subset?]

# Vector spaces WORKSHEET → SOLUTIONS

10/21/03

10/7/16

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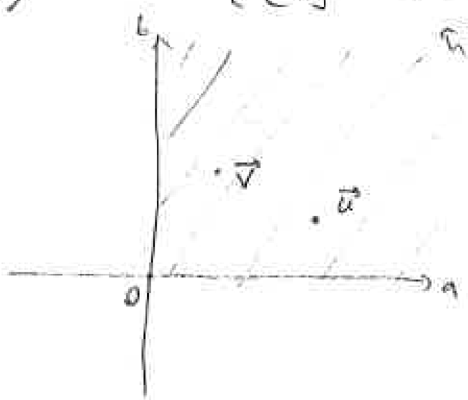
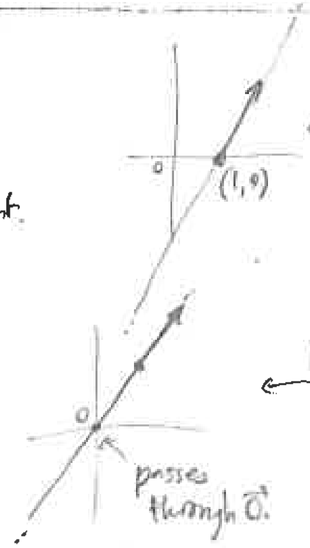
simply means, all points in the first quadrant.




Is it a subset? (T/F)
Property a) (T/F)
b) (T/F)
c) (T/F) ← since $c<0$ kicks you out of $H$.

so, is $H$ a subspace? (Y/N)

Hint: properties must be true for all scalars $c$, vectors $\vec{u},\vec{v}$.




picture

B) $H$ = all vectors of the form $\begin{bmatrix}1+a\\2a\end{bmatrix}$ where $a$ is a free parameter.

subset T/F, a) T/F, b) T/F, c) T/F

there's no $a$ so that $\begin{bmatrix}1+a\\2a\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

picture

C) $H$ = all vectors of the form $\begin{bmatrix}1+a\\2+2a\end{bmatrix}$

a) $a=-1$ makes $\begin{bmatrix}0\\0\end{bmatrix}$ ✓

b) notice set is same as $\begin{bmatrix}t\\2t\end{bmatrix}$ where $t=1+a$, so $\begin{bmatrix}s\\2s\end{bmatrix}+\begin{bmatrix}t\\2t\end{bmatrix}=\begin{bmatrix}s+t\\2(s+t)\end{bmatrix}$ ✓

c) $c\begin{bmatrix}t\\2t\end{bmatrix}=\begin{bmatrix}ct\\2(ct)\end{bmatrix}$ so is in same form

D) $H=\mathbb{R}^1$ ie one-component vectors.

[Hint: is it a subset?] → no!

⇒ not a subspace of $\mathbb{R}^2$. (is a subspace of $\mathbb{R}^1$, though)