—— SOLUTIONS —— Barnett

Your name:

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Math 22 Fall 2016, Midterm 2, Wed Oct 26

Please show your work. No credit is given for solutions without work or justification.

1. [8 points] Compute the determinants of the matrices in (a) and (b) (in each case there is a way that is quite quick).

3pts. (a) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 3 \\ 1 & -7 & 2 & 5 \\ 4 & 9 & 3 & 1 \end{bmatrix}$ ← use this row: sign -1

$$\det = -1 \begin{vmatrix} \cancel{0} & \cancel{2} & \cancel{0} \\ 1 & \cancel{-7} & 2 \\ 4 & \cancel{9} & 3 \end{vmatrix} = (-1)(-2)\underbrace{\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix}}_{3-8=-5}$$

$$= -10.$$

could also row reduce, but has lots of swaps

3pts (b) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 3 & 5 & 4 \end{bmatrix}$ Use row reduction:

$$\det = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 2 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{vmatrix}$$

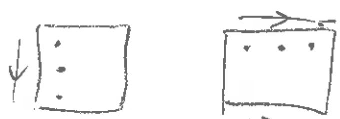
upper-triangular ⇒ product of diag. entries.

$$= -6$$

2pts. (c) Explain why if A is a 3×3 matrix, $\det A = \det A^T$.

First note the 2×2 case: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$ so not changed by transpose.

for 3×3, Use cofactor expansion: Going down 1st col. of A is same as going along 1st row of A^T.



A A^T 1

In each case the 2×2 cofactor blocks are transposed, so the same.

Also, det doesn't depend on only diag entries of A!

[Saying A & A^T reduce to same EF isn't true if det=0, & doesn't show values on pivots are equal; they are not!]

2. [9 points] Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

2pts. (a) Find (and simplify) the characteristic polynomial for A.

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)\Big[\underbrace{(1-\lambda)(1-\lambda)-1}_{\lambda^2-2\lambda}\Big] \overbrace{-\,1(\cancel{1}-\lambda-\cancel{1}) + 1(\cancel{1}-\cancel{1}+\lambda)}^{2\lambda}$$

$$= -\lambda\big((1-\lambda)(2-\lambda)-2\big) = +\lambda\big(\lambda(3-\lambda)\big) = \lambda^2(3-\lambda)$$

5pts. (b) Find the eigenvalues of A with their multiplicities. For each, give a basis for its eigenspace.

Eigenvalues are Roots of (a), are 0 (twice) multiplicity 2, 3 multiplicity 1.

$\boxed{\lambda = 0:}$ find Nul A:

$$A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

free: x_2 x_3

so $x_1 = -x_2 - x_3$
$x_2 = x_2$
$x_3 = x_3$

basis for eigenspace is $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\boxed{\lambda = 3:}$ find Nul $(A-3I)$:

x_3 free

$$A - 3I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ & & \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ & 1 & -1 \\ & & \end{bmatrix}$$

so $x_1 = +x_3$
$x_2 = x_3$
$x_3 = x_3$

$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, eigenvector.

2pts. (c) Evaluate $A^4 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = A^4\vec{x} = \lambda^4\vec{x} = 3^4\vec{x} = 81\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 81 \\ 81 \\ 81 \end{bmatrix}$

3. [9 points] Define the set of vectors $H = \left\{ \begin{bmatrix} a+b+2c \\ -b-c \\ 2a+b+3c \end{bmatrix} : \quad a,b,c \text{ real} \right\}$.

2pts (a) Explain why H is a vector space (you may use results from class).

$H = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \right\}$ and any Span is a subspace, hence a vector space.

3pt (b) Find a basis for H.

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 2 & 1 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

1pt (c) Is $H = \mathbb{R}^3$? No, since the 3 vectors do not span $\mathbb{R}^3$.

2pts (d) Each vector in H is a linear combination of the linearly independent standard basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. Are these vectors a basis for H, and why?

No. [Similar to the homework question in HW5 #2a, and 4.3.25]

$\vec{e}_1, \vec{e}_2, \vec{e}_3$ are not even in H. Their span is $\mathbb{R}^3$, which contains H but is not equal to H.

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ etc

1pt (e) For what p is H isomorphic to $\mathbb{R}^p$? (no explanation needed here)

$p=2$, since $\dim H = 2$

4. [8 points]

3pts (a) Is the set $V = \left\{ \begin{bmatrix} 2a+1 \\ a+1 \end{bmatrix} \ : \ a \text{ real} \right\}$ a vector space? Prove your answer.

setting $\begin{bmatrix} 2a+1 \\ a+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ gives $a = -\frac{1}{2}$ & $a = -1$

$\Rightarrow$ impossible

The $\vec{0}$ of $\mathbb{R}^2$ not in V. $\Rightarrow$ not a vector space.

$m \times n$, say. set of $\vec{x}$ s.t. $A\vec{x} = \vec{0}$.

3pts (b) Let A be any matrix. Then is the set Nul A a vector space? Prove your answer.

3 tests: (since it's a subset of $\mathbb{R}^n$)

a) $\vec{0}$ is in Nul$A$ since $A\vec{x} = \vec{0}$ always has trivial solution.

b) let $\vec{u}, \vec{v}$ be in Nul$A$, then is $\vec{u} + \vec{v}$ in Nul$A$?
Well, $A\vec{u} = \vec{0}$ & $A\vec{v} = \vec{0}$, adding gives $A\vec{u} + A\vec{v} = \vec{0}$
by linearity $\Rightarrow$ $A(\vec{u} + \vec{v}) = \vec{0}$, so $\vec{u} + \vec{v}$ is in Nul A.

c) Let $\vec{u}$ be in Nul$A$, so $A\vec{u} = \vec{0}$, and c be any scalar.
Mult. by c to get $c(A\vec{u}) = \vec{0}$, ie $A(c\vec{u}) = \vec{0}$ so $c\vec{u}$ in Nul$A$.

2pts. (c) If all solutions to a homogeneous 4×5 linear system are multiples of one nontrivial vector, then must the linear system be consistent whatever constants are chosen for the right-hand side? Explain. $\Rightarrow$ Yes.

$m \times n$ in matrix A, not augmented matrix!

A: $m=4$, $n=5$

dim Nul A = 1 since "multiples of one nontrivial vec."

By the Rank Theorem, $\underset{4}{\text{rank } A} + \underbrace{\text{dim Nul } A}_{1} = \underset{5}{n}$

So there's a pivot in every row,

so, consistent for every right-hand side, yes.

+1 → BONUS: Let A be a $m \times n$ matrix with Nul $A = \mathbb{R}^n$. What can you prove about A ?

A has all entries zero, ie. the zero matrix. Proof:

rank A = n - dim Nul A = $n - n$ = 0, so no pivots.

5. [8 points]

2pts (a) Give the definition of a set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ being a *basis* for a vector space V.

i) The set $\{\vec{v}_1, \ldots \vec{v}_n\}$ are Linearly independent.

ii) V is precisely equal to $\text{span}\{\vec{v}_1, \ldots \vec{v}_n\}$

3pts (b) Show that $\mathfrak{B} = \{t^2+1, t-2, t+3\}$ is a basis for $\mathbb{P}_2$.

By isomorphism of $\mathbb{P}_2$ to $\mathbb{R}^3$ via the coord map using standard basis $\{1, t, t^2\}$, use coords:

Are $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}-2\\1\\0\end{bmatrix}, \begin{bmatrix}3\\1\\0\end{bmatrix}$ a basis for $\mathbb{R}^3$? reduce the matrix from stacking as columns

$$\sim \begin{bmatrix}1 & 0 & 0\\ & -2 & 3\\ & 1 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0\\ & 1 & 1\\ & & 5\end{bmatrix}$$

EF, full rank (3 pivots) $\Rightarrow$ cols. are a basis for $\mathbb{R}^3$. $\square$

3pts (c) Let $\mathbf{v} = \underbrace{8t^2 - 4t + 6}$. Find its coordinate vector $[\mathbf{v}]_{\mathfrak{B}}$ relative to $\mathfrak{B}$ in part (b).

RHS is coords of $\vec{v}$ in std. basis.

Solve 3×3 lin sys:

$$\left[\begin{array}{ccc|c}1 & -2 & 3 & 6\\ 0 & 1 & 1 & -4\\ 1 & 0 & 0 & 8\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 0 & 0 & 8\\ & -2 & 3 & -2\\ & 1 & 1 & -4\end{array}\right]$$

$$\sim \begin{bmatrix}1 & 0 & 0 & 8\\ & 1 & 1 & -4\\ & & 5 & -10\end{bmatrix} \sim \left[\begin{array}{ccc|c}1 & & & 8\\ & 1 & & -2\\ & & 1 & -2\end{array}\right]$$

$$[\vec{v}]_B = \begin{bmatrix}8\\-2\\-2\end{bmatrix}$$

answer is in $\mathbb{R}^3$, not a poly!

6. [8 points] In this question only, no working is needed; just circle T or F.

(a) T / F: Row reduction of a square matrix preserves its eigenvalues.

No! messes them up.

(b) T / F: If the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ span a vector space V, then dim $V = p$.

would also require L. I.

(c) T / F: If A and B are row-equivalent, then rank A = rank B.

since by Rank Thm, rank A = dim Row A & Row space preserved.

(d) T / F: If A is an $n \times (n-1)$ matrix and rank $A = n - 2$, then dim Nul $A = 2$.

Rank-Nullity Theorem. $n-1 = \text{rank} A + \dim(\text{Nul} A)$

So $\dim(\text{Nul} A) = n-1-(n-2) = 1$

(e) T / F: For sufficiently small positive ϵ the computer will report the rank of the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1+\epsilon \end{bmatrix}$ as one.

yes, eg $\epsilon < 10^{-16}$ in usual computers.

(f) T / F: $\mathbb{R}^6$ is a subspace of $\mathbb{R}^7$.

(g) T / F: The matrix $\begin{bmatrix} -7 & -5 \\ 10 & 5 \end{bmatrix}$ has no real eigenvalues.

$\det\left(\begin{bmatrix} -7-\lambda & -5 \\ 10 & 5-\lambda \end{bmatrix}\right) = (-7-\lambda)(5-\lambda)+50 = \lambda^2+2\lambda+15$ $\quad \lambda = \frac{-2 \pm \sqrt{4-60}}{2} = \frac{-2 \pm \sqrt{-56}}{2}$

(h) T / F: The subset of continuous functions on $[0, 1]$ with $\int_0^1 f(t)dt = 0$ is a subspace of the set of continuous functions on $[0, 1]$.

① $\int_0^1 0 \, dt = 0$

② $\int_0^1 f_1(t) + f_2(t) \, dt = \int_0^1 f_1(t)dt + \int_0^1 f_2(t)dt = 0$

③ $\int_0^1 c f(t) = c \int_0^1 f(t) = 0$