Math 22 Final Exam Practice Problems

NOTE: In no sense should this collection of problems be construed as representative of the actual exam. These are simply some problems left over from the preparation of the exam or from previous exams which should serve to indicate the general level of expectation.

Problem 1. Find the values of c and d so that the matrix

$$B = \begin{pmatrix} 5 & 1 \\ c & d \end{pmatrix}$$

has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$. [*Hint:* What conditions ensure that $\operatorname{Nul}(B - \lambda_1 I)$ and $\operatorname{Nul}(B - \lambda_2 I)$ are non-zero?]

Problem 2. If a, b, c are real numbers with $a \neq c$, let

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

a. Diagonalize A.

b. Compute the four entries of A^{1000} in terms of a, b, c.

Problem 3.

a. Let W be a subspace of $\mathbb{R}^n$. Show that if $\mathbf{x} \in W$ and $\mathbf{x} \in W^\perp$ then $\mathbf{x} = \mathbf{0}$. [*Hint:* What is $\mathbf{x} \cdot \mathbf{x}$?]

b. Let A be an $m \times n$ matrix. Show that if $A^T A\mathbf{x} = \mathbf{0}$ then $A\mathbf{x} \in (\operatorname{Col} A)^\perp$. Use part (a) to conclude that $A\mathbf{x} = \mathbf{0}$.

c. Part (b) shows that $\operatorname{Nul} A^T A = \operatorname{Nul} A$. Use this and the rank theorem to show that $\operatorname{rank} A^T A = \operatorname{rank} A$.

Problem 4. Let

$$A = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 0 & 9 & 2 \end{pmatrix}$$

a. Find an eigenvalue/eigenvector pair for A. It is not necessary to compute the characteristic polynomial of A or perform any row reduction.

b. Show that $\det A = 0$. Conclude that 0 is an eigenvalue of A.

c. Use the information gained from parts (a) and (b) to diagonalize A.

Problem 5. Let A be an $n \times n$ matrix with linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Suppose that B is $n \times n$ and has the same eigenvectors with corresponding eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$. Show that $AB = BA$. [*Hint:* Use the given information to diagonalize A and B.]

Problem 6. Compute the determinant of the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 3 & 2 & 1 & 2 \end{pmatrix}$$

using the method of your choice.

Problem 7. Use any technique that you prefer to compute

$$\begin{vmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{vmatrix}.$$

Is the matrix inside the determinant invertible?

Problem 8. Let M be the singular matrix

$$M = \begin{pmatrix} a & b & c & d \\ a & b & c & d \\ a & b & c & d \\ a & b & c & d \end{pmatrix}.$$

Show that $\det(M + I) = 1 + a + b + c + d$. Using a cofactor expansion is probably not the best idea.

Problem 9. Given two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$, what value of the scalar x minimizes the quantity $||\mathbf{b} - x\mathbf{a}||$?

Problem 10. Suppose that the matrices A and B have *the same column space* (they may have different columns, though!).

a. Do A and B necessarily have the same number of pivots?

b. Do A and B necessarily have the same nullspace?

c. If A is invertible, can you conclude that B is invertible as well?

Problem 11. Suppose that the 3×3 matrix A has eigenvalues $\lambda_1 = 0$, $\lambda_2 = -3/4$ and $\lambda_3 = 1/2$. Let $\mathbf{u}_0$ be a vector in $\mathbb{R}^3$ and define the sequence of vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$ by $\mathbf{u}_{k+1} = A\mathbf{u}_k$. Is there a vector $\mathbf{v}$ so that $\mathbf{u}_k \to \mathbf{v}$ as $k \to \infty$? If so, does $\mathbf{v}$ depend on $\mathbf{u}_0$?