1. **(True or False?)** Label each statement as "true" or "false" (**you must write the whole word**). A statement is considered true only if it is always true; otherwise it is false.

(a) [2 points] FALSE If events A, B, C are pairwise independent then they are collectionwise independent.

(b) [2 points] FALSE If events A, B, C are collectionwise independent then they are pairwise independent.

(c) [2 points] TRUE Let A, B, C be pairwise disjoint events with $A \cup B \cup C = \Omega$. Then, if $F \subset \Omega$,

$$P(F) = P(A \cap F) + P(B \cap F) + P(C \cap F).$$

(d) [2 points] FALSE If $\{a_n\}, \{b_n\}$ are sequences of numbers with $a_n \sim b_n$, then $a_n^n \sim b_n^n$.

(e) [2 points] FALSE For a finite experiment, the average value and expected value of a random variable are equal.

(f) [2 points] FALSE If X and Y are random variables with $E(XY) = E(X)E(Y)$ then X and Y are independent.

(g) [2 points] TRUE If A^c and B are independent, then so are A and B.

2. [8 points] Suppose that every student in the class determines his or her answers to the seven questions in the True/False section of this exam by flipping a coin. What is the probability that there is at least one exam with two or fewer correct answers? You may assume that there are 41 students in the class and **you do not need to get a numerical answer** (just write the expression that would yield the correct answer).

Probability that a given student gets 2 or fewer Correct $= \underbrace{\binom{7}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^7 + \binom{7}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^6 + \binom{7}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^5}_{\text{i.e. } \sum_{i=0}^{2}\binom{7}{i}\left(\frac{1}{2}\right)^7}$

So $1 - \sum_{i=0}^{2}\binom{7}{i}\left(\frac{1}{2}\right)^7$ is the probability that a given student has at least 3 Correct answers.

Then $\left(1 - \sum_{i=0}^{2}\binom{7}{i}\left(\frac{1}{2}\right)^7\right)^{41}$ is the probability that <u>every</u> student in the class has at least 3 Correct answers.

Thus, $1 - \left(1 - \sum_{i=0}^{2}\binom{7}{i}\left(\frac{1}{2}\right)^7\right)^{41}$ is the probability that at least one student has 2 or fewer correct.

3. [8 points] A student is applying for college and would love to spend the next four years living in the Upper Valley, so she limits her search to Lebanon College and Dartmouth. She estimates that she has a probability of .5 of being accepted at Lebanon College and .3 of being accepted at Dartmouth. She further estimates the probability that she will be accepted by both is .2. What is the probability that she is accepted by Lebanon College if she is accepted by Dartmouth?

Let D be the event "accepted by Dartmouth."

Let L be the event "accepted by Lebanon College."

Then $P(D) = 0.3$

$P(L) = 0.5$

$P(L \cap D) = 0.2.$

So $P(L|D) = \dfrac{P(L \cap D)}{P(D)} = \dfrac{0.2}{0.3} = \boxed{\dfrac{2}{3}}$

4. [10 points] Prove the following binomial identity: $\binom{2n}{n} = \sum_{j=0}^{n} \binom{n}{j}^2$.

LHS:

Consider an urn with n red balls and n blue balls (i.e. 2n total balls). There are $\binom{2n}{n}$ ways of choosing any subset of n balls.

RHS:

We could think of choosing balls from 2 separate urns, one containing n red balls and one containing n blue balls. Choosing j red balls and n−j blue balls gives us a total of n. There are $\binom{n}{j}\binom{n}{n-j}$ ways to do this.

Note: In class, we showed that $\binom{n}{j} = \binom{n}{n-j}$, so $\binom{n}{j}\binom{n}{n-j} = \binom{n}{j}^2$.

Thus, to count all possible combinations, we must sum over j, i.e.

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \sum_{j=0}^{n} \binom{n}{j}^2.$$

* Since RHS and LHS are counting the same thing, they must be equal!

5. Suppose that n people have their hats returned at random, after checking them with a disorganized hat check boy. Let $X_i = 1$ if the i^{th} person gets his or her own hat back and 0 otherwise. Let $X = \sum_{i=1}^{n} X_i$. Then X is the total number of people who get their own hats back. Show that:

(a) [3 points] $E(X_i^2) = \frac{1}{n}$.

Note : $P(X_i = 1) = \frac{(n-1)!}{n!} = \frac{1}{n}$ ← # of ways of permuting the other $n-1$ hats if the ith hat is "fixed."

Thus, $E(X_i^2) = 1^2 \cdot P(X_i = 1) + 0^2 \cdot P(X_i = 0)$
$= 1^2 \cdot \frac{1}{n} = \boxed{\frac{1}{n}}$.

(b) [3 points] $E(X_i \cdot X_j) = \frac{1}{n(n-1)}$ for $i \neq j$.

$E(X_i \cdot X_j) = \sum_{\omega \in \Omega} X_i \cdot X_j(\omega) m(\omega) = 1 \cdot P(X_i = 1, X_j = 1) + 0 \cdot P(X_i = 0, X_j = 1) + 0 \cdot P(X_i = 1, X_j = 0) + 0 \cdot P(X_i = 0, X_j = 0) = \boxed{\frac{1}{n(n-1)}}$ ↑ from (*)

(*) [Note: $P(X_i = 1, X_j = 1) = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$]

(c) [3 points] $E(X^2) = 2$. (Hint: Use parts (a) and (b).)

$E(X^2) = E((\sum_{i=1}^{n} X_i)^2) = E(\sum_{i=1}^{n} X_i^2 + 2\sum_{i \neq j} X_i X_j)$

$= \sum_{i=1}^{n} E(X_i^2) + 2\sum_{i \neq j} E(X_i X_j)$ (by linearity of expectation)

$= \underbrace{n \cdot \frac{1}{n}}_{\text{from (a)}} + 2 \cdot \binom{n}{2} \cdot \underbrace{\frac{1}{n(n-1)}}_{\text{from (b)}} = 1+1 = \boxed{2}$

(d) [3 points] $V(X) = 1$.

$V(X) = E(X^2) - (E(X))^2$.

From (c), $E(X^2) = 2$.

Note : $E(X) = E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i) = n \cdot \frac{1}{n} = 1$

Thus, $V(X) = 2 - 1^2 = \boxed{1}$

6. [8 points] Homecoming Weekend is one of the most financially lucrative times of the year for the Dartmouth-Hitchcock Medical Center. Last year during Homecoming, $\frac{1}{8}$ of Dartmouth students experienced fire-related injuries (burns, dehydration and breathing problems due to smoke inhalation). Another $\frac{1}{4}$ of Dartmouth students suffered alcohol-related maladies (nausea, blackouts and alcohol poisoning). Dartmouth was also in the midst of a swine flu epidemic, which affected $\frac{1}{2}$ of all students. Of course, not all of these students had to go to the hospital and not all types of injuries were represented equally in the hospital waiting room. A reporter for the Upper Valley News (who spent the whole weekend in the waiting room obtaining the data for this study) found that the probability that a student with fire-related injuries wound up in the hospital was $\frac{1}{2}$. The probability that a student with alcohol-related maladies wound up in the hospital was $\frac{1}{4}$. The probability that a student with swine flu wound up in the hospital was $\frac{1}{8}$. What is the probability that a student had fire-related injuries, given that he spent part of Homecoming Weekend at DHMC?

$$\left.\begin{aligned} P(H \mid Fire) &= 1/2 \\ P(H \mid Alcohol) &= 1/4 \\ P(H \mid Flu) &= 1/8 \end{aligned}\right\}$$

$$P(H) = P(Fire) \cdot P(H \mid Fire) + P(Alcohol) \cdot P(H \mid Alcohol) + P(Flu) \cdot P(H \mid Flu)$$

$$= \frac{1}{8} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{8}$$

$$= \frac{3}{16}$$

Bayes' Theorem $\Rightarrow P(Fire \mid H) = \dfrac{P(H \mid Fire) \cdot P(Fire)}{P(H)}$

$$= \frac{1/2 \cdot 1/8}{3/16} = \boxed{\frac{1}{3}}$$

7. [10 points] Prove that if X and Y are independent random variables then $E(XY) = E(X)E(Y)$.

Let $\Omega_X = \{x_1, x_2, \ldots\}$ be the sample spaces
$\Omega_Y = \{y_1, y_2, \ldots\}$ for X & Y (respectively).

* To make the notation simple, we will assume that X & Y are numerically-valued random variables.

Then $$E(X \cdot Y) = \sum_j \sum_k x_j y_k P(X = x_j, Y = y_k)$$
(by definition of $E(X \cdot Y)$)

$$= \sum_j \sum_k x_j y_k P(X = x_j) \cdot P(Y = y_k)$$
(Since X, Y are independent)

$$= \left(\sum_j x_j P(X = x_j) \right) \cdot \left(\sum_k y_k P(Y = y_k) \right)$$
(Since we can pull the terms without k's in front of the sum over k)

$$= E(X) \cdot E(Y)$$ (by definition of $E(X)$ & $E(Y)$)

8. Homer Simpson has just been made leader of The Stonecutters, an elite secret society! One of the rules of The Stonecutters is that the leader is infallible. Accordingly, when Homer plays poker with his fellow Stonecutters, he always has to win. In order to justify his wins, the Stonecutters make up elaborate names for the hands that he is dealt. On one occasion, Homer is dealt a 3, 6, 10, Jack, King, which the Stonecutters call the "Royal Sampler." Assume that these cards come from at least two different suits (so that the "Royal Sampler" is distinct from a Flush).

   (a) [4 points] In general, what is the probability of drawing the "Royal Sampler?"

$$P(\text{Royal Sampler}) = \frac{\binom{4}{1}^5 - \binom{4}{1}}{\binom{52}{5}}$$

# of Royal Samplers that are also Flushes

   (b) [6 points] In the same round in which Homer drew the "Royal Sampler," the police chief Wiggum drew a pair of kings, his favorite bartender Moe drew a pair of aces, and his coworkers Lenny and Carl drew three 2's and a pair of 10's (respectively). Out of the five players in this round, whose hand "should" be worth the most? Defend your choice by comparing the probabilities of the various hands and/or citing results from the lectures and homework.

$$P(\text{3-of-a-kind}) = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2}{\binom{52}{5}}$$

* We learned in class that $P(\text{3-of-a-kind}) < P(\text{pair})$, so 3-of-a-kind should be worth more than a pair.

* To compare $P(\text{3-of-a-kind})$ w/ $P(\text{Royal Sampler})$, we just need to compare the numerators:

$$\binom{4}{1}^5 - \binom{4}{1} < \binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2 \quad \therefore \text{Homer should win!}$$

(c) [6 points] What is the expected number of kings in a poker hand? To receive full credit, you must define the random variables that you use and show each step of the computation.

Let $X_i = \begin{cases} 1 & \text{if the } i^{th} \text{ card is a king} \\ 0 & \text{otherwise} \end{cases}$

Then $X = \sum_{i=1}^{5} X_i$ = # of kings in a poker hand

$$E(X) = E\left(\sum_{i=1}^{5} X_i\right) = \sum_{i=1}^{5} E(X_i) \quad \text{(by linearity of expectation)}$$

$$= \sum_{i=1}^{5} 1 \cdot \underbrace{P(X_i=1)}_{= 1/13 \text{, since there are 4 kings in a deck of 52 cards}} + 0 \cdot P(X_i=0)$$

$$= \sum_{i=1}^{5} 1/13 = \boxed{5/13}$$

(d) [2 points] Suppose that, instead of just having a pair of kings, Wiggum realizes that he actually has four kings in his hand (d'oh!). Given that Wiggum has four kings, what is the probability that Homer has the "Royal Sampler?"

The probability is 0 - Wiggum already has all of the kings!

9. **Short Answer** (no need to justify your steps - on each problem, credit will be awarded on an "all or nothing" basis)

   (a) [4 points] Alice and Bob each flip n coins. What is the probability that they get an equal number of heads? Give a simple closed-form expression.

   $$\boxed{\frac{\binom{2n}{n}}{2^{2n}}} \quad \left(\text{or } \frac{\sum_{j=0}^{n} \binom{n}{j}^2}{2^{2n}}\right.$$

   – from #4, we know that these expressions are equivalent)

   (b) [4 points] Count the number of n-digit integers with *nondecreasing* digits (for example, if $n = 4$, then 2357 and 2337 should be included in your count, but 3434 should not be included). Your answer should be a binomial coefficient. Do not evaluate!

   "Stars & Bars" $\rightarrow \binom{9+n-1}{n} = \binom{8+n}{n}$

   (c) [4 points] Suppose that you are playing a game in which you flip a coin until you get heads. If the coin lands heads for the first time on the n^{th} flip, you earn 2^n dollars. If X is the number of dollars that you win by playing this game, what is $E(X)$?

   $\boxed{\infty}$ (This is the St. Petersburg Paradox)

10. [2 points (bonus)] Give a very simple proof, without using Stirling's formula, that $\ln(n!) \sim n \ln n$.

$$\ln(n!) = \ln\left(\prod_{i=1}^{n} i\right) = \sum_{i=1}^{n} \ln(i)$$

from logarithm rules
(i.e. $\ln(ab) = \ln(a) + \ln(b)$)

Note: $\displaystyle\sum_{i=1}^{n} \ln(i) \approx \int_1^n \ln(x)\,dx$ (from calculus)

$$= n\ln(n) - n$$

$$\sim n\ln(n). \quad \left(\text{Since } \lim_{n\to\infty} \frac{n\ln n - n}{n \ln n}\right.$$

$$= \lim_{n\to\infty} 1 - \frac{n}{n\ln n}$$

$$= \lim_{n\to\infty} 1 - \frac{1}{\ln n}$$

$$\left.= 1\right)$$

This page is for scratch work.

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Stirling's Formula:

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$$

Bayes' Theorem:

$$P(H_i \mid E) = \frac{P(H_i)P(E \mid H_i)}{P(E)},$$

where $P(E) = \sum_{k=1}^{m} P(H_k)P(E \mid H_k)$

Variance Definition: Let $E(X) = \mu$. Then:

$$V(X) = E((X - \mu)^2),$$

i.e.

$$V(X) = \sum_{\omega \in \Omega} (X(\omega) - \mu)^2 m(\omega)$$

Variance "Trick":

$$V(X) = E(X^2) - \mu^2$$

Variance Rules: Let X, Y be two random variables. Let c be a constant. Then:

$$V(c) = 0$$

$$V(X + c) = V(X)$$

$$V(cX) = c^2 V(X)$$

If X and Y are independent then:

$$V(X + Y) = V(X) + V(Y).$$

Standard Deviation:

$$\sigma(X) = \sqrt{V(X)}$$