61 a) $x(t) = x_0 + v_0 t \qquad y(t) = 0 \qquad z(t) = 0$

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} = (x_0 + v_0 t)\hat{i}$$

b) $a_x = \frac{dv_x}{dt} = a_0 \implies v_x = \int a_0\,dt = a_0 t + c_1$

but $v_0 = v_x(0) = a_0(0) + c_1 = c_1$, so $v_x = a_0 t + v_0$

$$v_x = \frac{dx}{dt} = a_0 t + v_0 \implies x = \int (a_0 t + v_0)\,dt = \tfrac{1}{2} a_0 t^2 + v_0 t + c_2$$

but $x_0 = x(0) = \frac{1}{2} a_0 (0)^2 + v_0(0) + c_2 = c_2$, so $x(t) = \frac{1}{2} a_0 t^2 + v_0 t + x_0$

$$y(t) = 0 \;\&\; z(t) = 0$$

$$\implies \vec{r}(t) = \left(\tfrac{1}{2} a_0 t^2 + v_0 t + x_0\right)\hat{i}$$

c) Let $v_x = v_y = v_0$

$x(t) = x_0 + v_0 t$

$y(t) = y_0 + v_0 t$

$z(t) = 0$

$$\implies \vec{r}(t) = (x_0 + v_0 t)\hat{i} + (y_0 + v_0 t)\hat{j}$$

d) Let $v_x = v_y = v_z = v_0$

$$x(t) = x_0 + v_0 t$$

$$y(t) = y_0 + v_0 t$$

$$z(t) = z_0 + v_0 t$$

$$\Rightarrow \vec{r}(t) = (x_0 + v_0 t)\hat{i} + (y_0 + v_0 t)\hat{j} + (z_0 + v_0 t)\hat{k}$$

62 x-comp: $x(t) = x_0 + v_x t$

y-comp: $y(t) = 0$

z-comp: $a_z = \frac{dv_z}{dt} = -g \Rightarrow v_z = \int -g\,dt = -gt + c_1$

but $-v_{0z} = v_z(0) = -g(0) + c_1$, so $c_1 = -v_{0z}$

$$v_z = \frac{dz}{dt} = -gt - v_{0z} \Rightarrow z = \int(-gt - v_{0z})dt = -\frac{1}{2}gt^2 - v_{0z}t + c_2$$

but $z_0 = z(0) = -\frac{1}{2}g(0)^2 - v_{0z}(0) + c_2$, so $c_2 = z_0$

$$z(t) = -\frac{1}{2}gt^2 - v_{0z}t + z_0$$

$$\Rightarrow \vec{r}(t) = (x_0 + v_x t)\hat{i} + (-\frac{1}{2}gt^2 - v_{0z}t + z_0)\hat{k}$$

$$= (x_0 + v_x t)\hat{i} - (\frac{1}{2}gt^2 + v_{0z}t - z_0)\hat{k}$$

This object follows a parabolic path in the xz-plane.

63 $\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k} = \cos(3t)\,\hat{i} + \sin(3t)\,\hat{j}$

This object moves in a circle in the xy-plane.

64 $\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k} = \cos(3t)\,\hat{i} + \sin(3t)\,\hat{j} + t\,\hat{k}$

This object moves in a helix centered on the z-axis.

65 $$\vec{r}(t) = \cos(3t)\,\hat{i} + \sin(3t)\,\hat{j} + t\,\hat{k}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d}{dt}(\cos(3t))\hat{i} + \frac{d}{dt}(\sin(3t))\hat{j} + \frac{d}{dt}(t)\hat{k}$$

$$= -3\sin(3t)\,\hat{i} + 3\cos(3t)\,\hat{j} + 1\,\hat{k}$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d}{dt}(-3\sin(3t))\hat{i} + \frac{d}{dt}(3\cos(3t))\hat{j} + \frac{d}{dt}(1)\,\hat{k}$$

$$= -9\cos(3t)\,\hat{i} - 9\sin(3t)\,\hat{j} + 0\,\hat{k}$$

$$|\vec{v}| = \sqrt{(-3\sin(3t))^2 + (3\cos(3t))^2 + (1)^2} = \sqrt{9(\sin^2(3t) + \cos^2(3t)) + 1}$$

$$= \sqrt{9+1} = \sqrt{10}$$

$$\vec{T} = \frac{1}{|\vec{v}|}\vec{v} = \frac{1}{\sqrt{10}}\left(-3\sin(3t)\,\hat{i} + 3\cos(3t)\,\hat{j} + 1\,\hat{k}\right)$$

$$= -\frac{3}{\sqrt{10}}\sin(3t)\,\hat{i} + \frac{3}{\sqrt{10}}\cos(3t)\,\hat{j} + \frac{1}{\sqrt{10}}\,\hat{k}$$

$$\frac{d\vec{T}}{dt} = \frac{d}{dt}\left(-\frac{3}{\sqrt{10}}\sin(3t)\right)\hat{i} + \frac{d}{dt}\left(\frac{3}{\sqrt{10}}\cos(3t)\right)\hat{j} + \frac{d}{dt}\left(\frac{1}{\sqrt{10}}\right)\hat{k}$$

$$= -\frac{9}{\sqrt{10}}\cos(3t)\,\hat{i} - \frac{9}{\sqrt{10}}\sin(3t)\,\hat{j} + 0\,\hat{k}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\left(-\frac{9}{\sqrt{10}}\cos(3t)\right)^2 + \left(-\frac{9}{\sqrt{10}}\sin(3t)\right)^2 + 0^2}$$

$$= \sqrt{\frac{81}{10}\left(\cos^2(3t) + \sin^2(3t)\right)} = \sqrt{\frac{81}{10}} = \frac{9}{\sqrt{10}}$$

$$s(t) = \int_0^t |\vec{v}|\,dt' = \int_0^t \sqrt{10}\,dt' = \sqrt{10}\,t$$

The acceleration vector points from the object perpendicularly toward the z-axis.

66 $$\vec{r}(t) = (t^3+1,\ 4t^3-8)$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \left(\frac{d}{dt}(t^3+1),\ \frac{d}{dt}(4t^3-8)\right) = (3t^2,\ 12t^2)$$

$$|\vec{v}| = \sqrt{(3t^2)^2+(12t^2)^2} = \sqrt{9t^4+144t^4} = t^2\sqrt{153}$$

$$\vec{T} = \frac{1}{|\vec{v}|}\vec{v} = \frac{1}{t^2\sqrt{153}}(3t^2,\ 12t^2) = \left(\frac{3}{\sqrt{153}},\ \frac{12}{\sqrt{153}}\right)$$

$$s(t) = \int_0^t \sqrt{153}\, t'^2\, dt' = \sqrt{153}\left(\frac{1}{3}t'^3\Big|_0^t\right)$$

$$= \sqrt{153}\left(\frac{1}{3}t^3\right) = \frac{\sqrt{153}}{3}t^3$$

Since $\vec{T}$ is a constant (i.e. independent of time), the object moves in a straight line in the $\vec{T}$ direction.

67 $$\vec{r}(t) = (t,\ t^{3/2})$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \left(1,\ \frac{3}{2}t^{1/2}\right)$$

$$|\vec{v}| = \sqrt{(1)^2+\left(\frac{3}{2}t^{1/2}\right)^2} = \sqrt{1+\frac{9}{4}t}$$

$$s(t) = \int_0^t \sqrt{1+\frac{9}{4}t'}\, dt' \qquad \text{let } u = 1+\frac{9}{4}t' \Rightarrow du = \frac{9}{4}dt'$$

$$\Rightarrow s(t) = \int_1^{1+\frac{9}{4}t} \sqrt{u}\left(\frac{4}{9}du\right) = \frac{4}{9}\left(\frac{2}{3}u^{3/2}\Big|_1^{1+\frac{9}{4}t}\right)$$

$$= \frac{4}{9}\left(\frac{2}{3}\left(1+\frac{4}{9}t\right)^{3/2} - \frac{2}{3}(1)^{3/2}\right) = \frac{8}{27}\left(\left(1+\frac{4}{9}t\right)^{3/2} - 1\right)$$

68

$$\vec{v}(t) = (t, \cos t, \tan t)$$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \left(\int t\,dt, \int \cos t\,dt, \int \tan t\,dt\right)$$

$$= \left(\frac{1}{2}t^2 + c_1, \sin t + c_2, -\ln|\cos t| + c_3\right)$$

$$\Rightarrow \vec{r}(0) = \left(\frac{1}{2}(0)^2 + c_1, \sin(0) + c_2, -\ln|\cos(0)| + c_3\right)$$

$$= (c_1, c_2, c_3) = (1, 2, -1) \Rightarrow c_1 = 1, c_2 = 2, c_3 = -1$$

$$\Rightarrow \vec{r}\left(\frac{\pi}{4}\right) = \left(\frac{1}{2}\left(\frac{\pi}{4}\right)^2 + 1, \sin\left(\frac{\pi}{4}\right) + 2, -\ln\left|\cos\left(\frac{\pi}{4}\right)\right| - 1\right)$$

$$= \left(\frac{\pi^2}{8} + 1, \frac{\sqrt{2}}{2} + 2, -\ln\left(\frac{\sqrt{2}}{2}\right) - 1\right)$$

69

$$\vec{v}(t) = (\cos(t), \cos(2t), 3\sin(2t))$$

$$\Rightarrow \vec{r}(t) = \int \vec{v}(t)\,dt = \left(\int \cos(t)\,dt, \int \cos(2t)\,dt, \int 3\sin(2t)\,dt\right)$$

$$= \left(\sin(t) + c_1, \frac{1}{2}\sin(2t) + c_2, -\frac{3}{2}\cos(2t)\right)$$

The first term returns to its starting value at $t = \pi, 2\pi, 3\pi, \ldots$
The second term returns to its starting value at $t = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \ldots$
The third term returns to its starting value at $t = \pi, 2\pi, 3\pi, \ldots$

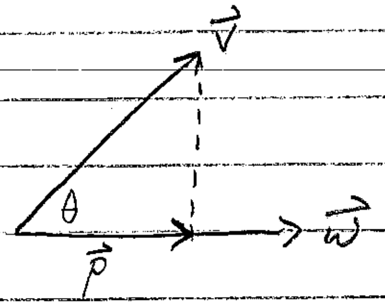
$\Rightarrow \vec{r}(t)$ returns to $\vec{r}_0$ for the first time at $t = \pi$.

70 $$\frac{d}{dt}(m\vec{v}) = \frac{d}{dt}\left(mv_x\hat{i} + mv_y\hat{j} + mv_z\hat{k}\right)$$

$$= m\frac{dv_x}{dt}\hat{i} + m\frac{dv_y}{dt}\hat{j} + m\frac{dv_z}{dt}\hat{k}$$

$$= m\left(\frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{z}\right) = m\frac{d\vec{v}}{dt}$$

72



$$\hat{w} = \frac{1}{|\vec{w}|}\vec{w}$$

$$\vec{p} = |\vec{v}|\cos\theta\ \hat{w}$$

$$= |\vec{v}||\hat{w}|\cos\theta\ \hat{w} \quad \text{since } |\hat{w}| = 1$$

$$= (\vec{v}\cdot\hat{w})\ \hat{w}$$

or $$\vec{p} = \left[\vec{v}\cdot\left(\frac{1}{|\vec{w}|}\vec{w}\right)\right]\frac{1}{|\vec{w}|}\vec{w} = \left(\frac{1}{|\vec{w}|^2}\vec{v}\cdot\vec{w}\right)\vec{w}$$

7 $\vec{F}_g = -mg\hat{j} \qquad \vec{F}_f = -\beta\vec{v} = -\beta v_x\hat{i} - \beta v_y\hat{j}$

Newton's 2^{nd} Law: $\vec{F}_g + \vec{F}_f = m\vec{a}$

$$\Longrightarrow \quad -mg\hat{j} - \beta v_x\hat{i} - \beta v_y\hat{j} = ma_x\hat{i} + ma_y\hat{j}$$

x-comp: $m a_x = m\frac{dv_x}{dt} = -\beta v_x$

$$\Longrightarrow \quad \frac{dv_x}{v_x} = -\frac{\beta}{m}\,dt$$

$$\Longrightarrow \quad \int\frac{dv_x}{v_x} = -\frac{\beta}{m}\int dt \quad \Longrightarrow \ln v_x = -\frac{\beta}{m}t + c_1$$

$$\Longrightarrow \quad v_x = e^{-\frac{\beta}{m}t + c_1} = A_1 e^{-\frac{\beta}{m}t}, \text{ where } A_1 = e^{c_1}$$

@ $t=0$, $v_x = V_0\cos\theta \Longrightarrow V_0\cos\theta = A\,e^{-\frac{\beta}{m}(0)} = A$

$$x = \int v_x\,dt = V_0\cos\theta\int e^{-\frac{\beta}{m}t}\,dt$$

$$= -\frac{mV_0}{\beta}\cos\theta\, e^{-\frac{\beta}{m}t} + c_2$$

@ $t=0$, $x=0 \Longrightarrow 0 = -\frac{mV_0}{\beta}\cos\theta\, e^{-\frac{\beta}{m}(0)} + c_2$

so $\quad c_2 = \frac{mV_0}{\beta}\cos\theta$

$$x(t) = -\frac{mV_0}{\beta}\cos\theta\left(e^{-\frac{\beta}{m}t} - 1\right)$$

y-comp: $\quad ma_y = m\frac{dv_y}{dt} = -\beta v_y - mg$

$$\Rightarrow \frac{m\,dv_y}{-\beta v_y - mg} = dt$$

$$\Rightarrow \int \frac{m\,dv_y}{-\beta v_y - mg} = \int dt$$

Let $u = -\beta v_y - mg$, $du = -\beta dv_y \Rightarrow dv_y = -\frac{1}{\beta} du$

$$\Rightarrow -\frac{m}{\beta}\int \frac{du}{u} = \int dt \Rightarrow -\frac{m}{\beta}\ln u = t + c_3$$

$$\Rightarrow \ln(-\beta v_y - mg) = -\frac{\beta}{m}(t + c_3)$$

$$\Rightarrow -\beta v_y - mg = e^{-\frac{\beta}{m}(t+c_3)} = Be^{-\frac{\beta}{m}t}, \text{ where } B = e^{-\frac{\beta c_3}{m}}$$

@ $t = 0$, $v_y = V_0 \sin\theta \Rightarrow -\beta V_0 \sin\theta - mg = Be^{-\frac{\beta}{m}(0)}$

$$\Rightarrow B = -\beta V_0 \sin\theta - mg$$

So, $-\beta v_y - mg = (-\beta V_0 \sin\theta - mg)e^{-\frac{\beta}{m}t}$

or $\quad v_y = \left(V_0 \sin\theta + \frac{mg}{\beta}\right)e^{-\frac{\beta}{m}t} - \frac{mg}{\beta}$

$$\Rightarrow \quad y(t) = \int v_y\,dt = \int\left[\left(V_0\sin\theta + \frac{mg}{\beta}\right)e^{-\frac{\beta}{m}t} - \frac{mg}{\beta}\right]dt$$

$$= -\frac{m}{\beta}\left(V_0\sin\theta + \frac{mg}{\beta}\right)e^{-\frac{\beta}{m}t} - \frac{mg}{\beta}t + c_4$$

@ $t=0,\ y=0 \Rightarrow 0 = -\frac{m}{\beta}\left(V_0 \sin\theta + \frac{mg}{\beta}\right)e^{-\frac{\beta}{m}(0)} - \frac{mg}{\beta}(0) + c_4$

So, $c_4 = \frac{m}{\beta}\left(V_0 \sin\theta + \frac{mg}{\beta}\right)$

$\Rightarrow \quad y(t) = -\frac{m}{\beta}\left(V_0 \sin\theta + \frac{mg}{\beta}\right)\left[e^{-\frac{\beta}{m}t} + 1\right] - \frac{mg}{\beta}t$

Whew!