ex 1191

(a) $\vec{F}(x,y,z) = (x,y,z)$; γ is the line segment from $(2,2,1)$ to $(3,3,5)$.

First, we need to parameterize γ.

γ is a straight line, so we can express it using the direction vector: $(3,3,5)-(2,2,1)=(1,1,4)$.

$\gamma(t) = (2,2,1) + t\cdot(1,1,4), 0 \le t \le 1.$

Now, the work done by $\vec{F}$ is

$$\int_\gamma \vec{F}(x,y,z)\cdot d\vec{r}(t). \quad (*)$$

... So we need expressions for $x, y, \& z$.

$x(t) = 2+t, \; y(t) = 2+t, \; \& \; z(t) = 1+4t.$

Also, $\frac{d\vec{r}(t)}{dt} = (1, 1, 4)$ (as you should expect.)

$$\Rightarrow (*) = \int_0^1 (2+t, 2+t, 1+4t) \bullet (dt, dt, 4dt)$$

$$= \int_0^1 [(2+t)dt + (2+t)dt + (1+4t)4dt]$$

$$= \int_0^1 (8+18t)dt = [8t + 9t^2]_0^1 = 8+9 = 17$$

ex 11.9 (b) Due to the similarity to part (a), I will show only the part that is different.

$$W = \int_{\Gamma} (z,x,y) \bullet (dt, dt, 4dt)$$

$$= \int_0^1 \left[(1+4t)dt + (2+t)dt + (2+t)\cdot 4dt \right]$$

$$= \int_0^1 (11 + 9t)dt = \left[11t + \tfrac{9}{2}t^2 \right]_0^1 = 11 + \tfrac{9}{2}$$

$$= \frac{31}{2}.$$

(c) In this part we are given the parameterization, but not the range of t.

Since it starts at $(1,0,0) = (\cos(t_i), \sin(t_i), t_i)$, t_i must be 0. Similarly, $(\cos(t_f), \sin(t_f), t_f) = (-1, 0, \pi) \Rightarrow t_f = \pi$. So $0 \le t \le \pi$.

Now, $d\vec{r}(t) = (-\sin(t)dt, \cos(t)dt, dt)$.

Thus $W = \int_0^{\pi} (z,x,y) \bullet d\vec{r}(t) = \int_0^{\pi} (t, \cos(t), \sin(t)) \bullet (-\sin(t), \cos(t), 1) dt$

ex 119 (c) ct'd

$$= \int_0^{\pi} \left(-t\sin(t) + \cos^2(t) + \sin(t)\right)dt$$

$$= \underbrace{-\int_0^{\pi} t\sin(t)\,dt}_{I_1} + \underbrace{\int_0^{\pi} \cos^2(t)\,dt}_{I_2} + \underbrace{\int_0^{\pi} \sin(t)\,dt}_{I_3}$$

I_1: Integration by parts: $u = t$, $du = dt$

$v = -\cos(t)$, $dv = \sin(t)\,dt$

So $I_1 = -\left(-t\cos(t) + \int \cos(t)\,dt\right)\Big|_0^{\pi}$

$$= -\left(-t\cos(t) + \sin(t)\right)\Big|_0^{\pi} = \left(t\cos(t) - \sin(t)\right)\Big|_0^{\pi}$$

$$= \left(\pi\cos(\pi) - \sin(\pi)\right) - \left(0\cdot\cancel{\cos(0)} - \cancel{\sin(0)}\right)$$

$$= \pi \cdot -1 - 0 = -\pi$$

used integration table (or use half angle formula)

$$I_2 = \left(\frac{1}{2}t + \frac{1}{4}(\sin(2t))\right)\Big|_0^{\pi}$$

$$= \left(\frac{1}{2}\pi + \frac{1}{4}\sin(2\pi)\right) - \left(\cancel{\frac{1}{2}\cdot 0 + \frac{1}{4}\sin(2\cdot 0)}\right)$$

$$= \frac{\pi}{2} + 0 = \frac{\pi}{2}$$

$$I_3 = -\cos(t)\Big|_0^{\pi} = -\cos(\pi) - (-\cos(0)) = -(-1) - (-1)$$

$$= 1 + 1 = 2.$$

$$\Longrightarrow W = -\pi + \frac{\pi}{2} + 2 = 2 - \frac{\pi}{2}$$

Ex 120] **proof:** Suppose the particle traverses a path $\vec{r}(t)$, with initial point $\vec{r}(0)$ & final point $\vec{r}(1)$.

The work done is $\int_{r} \hat{F}(x,y,z) \cdot d\vec{r}(t)$

$$= \int_0^1 (F_1, F_2, F_3) \cdot (d\vec{r}_1(t), d\vec{r}_2(t), d\vec{r}_3(t))$$

$$= \int_0^1 (F_1 d\vec{r}_1(t) + F_2 d\vec{r}_2(t) + F_3 d\vec{r}_3(t))$$

$$= \int_0^1 F_1 d\vec{r}_1(t) + \int_0^1 F_2 d\vec{r}_2(t) + \int_0^1 F_3 d\vec{r}_3(t)$$

$$= F_1 \int_0^1 d\vec{r}_1(t) + F_2 \int_0^1 d\vec{r}_2(t) + F_3 \int_0^1 d\vec{r}_3(t)$$

$$= F_1 \left(\vec{r}_1(t) \Big|_0^1\right) + F_2 \left(\vec{r}_2(t) \Big|_0^1\right) + F_3 \left(\vec{r}_3(t) \Big|_0^1\right)$$

Just to be nice

$$= \left[\vec{F} \cdot \vec{r}(t)\right]_0^1.$$

Since this last expression depends only on $\vec{F}$ and $\vec{r}(0)$, $\vec{r}(1)$, we are done.

QED

Ex [21]

$$\int_{\gamma} \vec{F} \cdot d\vec{x} = \int_{t_s}^{t_f} \vec{F}(x(t), y(t), z(t)) \cdot (dx(t), dy(t), dz(t))$$

$$= \int_{t_s}^{t_f} (F_1(x(t)), F_2(y(t)), F_3(z(t))) \cdot (dx(t), dy(t)\, dz(t))$$

$$= \int_{t_s}^{t_f} \Big(F_1(x(t))\, dx(t) + F_2(y(t))\, dy(t) + F_3(z(t))\, dz(t) \Big)$$

$$= \int_{t_s}^{t_f} F_1(x(t))\, dx(t) + \int_{t_s}^{t_f} F_2(y(t))\, dy(t) + \int_{t_s}^{t_f} F_3(z(t))\, dz(t)$$

$$= \int_{t_s}^{t_f} F_1(x(t)) \frac{dx(t)}{dt}\, dt + \int_{t_s}^{t_f} F_2(y(t)) \frac{dy(t)}{dt}\, dt + \int_{t_s}^{t_f} F_3(z(t)) \frac{dz(t)}{dt}\, dt$$

change of variables

$$= \int_{x_s}^{x_f} F_1(x)\, dx + \int_{y_s}^{y_f} F_2(y)\, dy + \int_{z_s}^{z_f} F_3(z)\, dz .$$

No, NOT the Fraternity!

$\boxed{A\mathcal{E}X\ 18}$ We should probably compile a table of derivatives for $\sin(x) = f(x)$.

$$
\begin{array}{ll}
f^{(0)}(x) = \sin(x) & f^{(0)}(0) = 0 \\
f^{(1)}(x) = \cos(x) & f^{(1)}(0) = 1 \\
f^{(2)}(x) = -\sin(x) \Rightarrow & f^{(2)}(0) = 0 \\
f^{(3)}(x) = -\cos(x) & f^{(3)}(0) = -1 \\
f^{(4)}(x) = \sin(x) & f^{(4)}(0) = 0 \\
\vdots & \vdots
\end{array}
$$

using this information!

Now, $P_{16}(x) = \sum_{i=0}^{16} \frac{f^{(i)}(0)}{i!}(x-0)^i = \sum_{i=0}^{16} \frac{f^{(i)}(0)\, x^i}{i!}$

$$= 0 + \frac{x^1}{1!} + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 - \frac{x^7}{7!} + 0 + \frac{x^9}{9!}$$

$$+ 0 - \frac{x^{11}}{11!} + 0 + \frac{x^{13}}{13!} + 0 - \frac{x^{15}}{15!} + 0.$$

(or, as a formula...)

$$= \sum_{j=0}^{7} \frac{(-1)^j x^{2j+1}}{(2j+1)!}.$$

AEX 19 1) Find $P_8(x)$ at $x=1$ for $f(x)=\ln(x)$.

OK. We need a formula for the derivatives of f.

$$f^{(0)}(x)=\ln(x) \qquad f^{(0)}(1)=\ln(1)=0$$
$$f^{(1)}(x)=x^{-1} \qquad f^{(1)}(1)=1$$
$$f^{(2)}(x)=-1\cdot x^{-2} \quad \Rightarrow \quad f^{(2)}(1)=-1$$
$$f^{(3)}(x)=2\cdot x^{-3} \qquad f^{(3)}(1)=2$$
$$f^{(4)}(x)=-6\cdot x^{-4} \qquad f^{(4)}(1)=-6$$

for $K>0$

$$\longrightarrow f^{(K)}(x)=(-1)^{K-1}\cdot(K-1)!\cdot x^{-K}, \quad f^{(K)}(1)=(-1)^{K-1}(K-1)!$$

Now, using the formula on pg. A29, we see that

$$P_8(x)=0+\sum_{K=1}^{8}\frac{(-1)^{K-1}(K-1)!\,(x-1)^K}{K!}=\sum_{K=1}^{8}\frac{(-1)^{K-1}\cancel{(K-1)!}(x-1)^K}{K\cdot\cancel{(K-1)!}}$$

1st term is 0, so start sum at K=1.

$$=\sum_{K=1}^{8}\frac{(-1)^{K-1}(x-1)^K}{K}$$

$$=\frac{(x-1)^1}{1}-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\frac{(x-1)^5}{5}$$

$$-\frac{(x-1)^6}{6}+\frac{(x-1)^7}{7}-\frac{(x-1)^8}{8}.$$

AEX 19] 2) Find $P_7(x)$ at $x=1$ for $g(x)=\frac{1}{x}$.

Since $\frac{d}{dx}\ln(x)=\frac{1}{x}$, we can recycle some of our work from part 1).

Specifically, $g^{(k)}(x)=f^{(k+1)}(x)=(-1)^{k+1-1}(k+1-1)!\,x^{-(k+1)}$.

$=(-1)^k k!\,x^{-(k+1)}$. $\Rightarrow g^{(k)}(1)=(-1)^k k!$.

Again, the formula yields $P_7(x)=\sum_{k=0}^{7}\frac{(-1)^k \cancel{k!}\,(x-1)^k}{\cancel{k!}}$

$=\sum_{k=0}^{7}(-1)^k(x-1)^k = (x-1)^0-(x-1)^1+(x-1)^2-(x-1)^3$

$+(x-1)^4-(x-1)^5+(x-1)^6-(x-1)^7$.

3) Derivative of 1st polynomial: $1-(x-1)+(x-1)^2-(x-1)^3$
$+(x-1)^4-(x-1)^5+(x-1)^6-(x-1)^7$.

So... YES

AEX 20] Find $P_n(x)$ at $x=1$ for $f(x)=x^2$ for $n=0,1,2,$ and 3.

Derivatives:

$$\begin{aligned} f^{(0)}(x) &= x^2 & & f^{(0)}(1) = 1 \\ f^{(1)}(x) &= 2x & \Longrightarrow\ & f^{(1)}(1) = 2 \\ f^{(2)}(x) &= 2 & & f^{(2)}(1) = 2 \\ f^{(3)}(x) &= 0 & & f^{(3)}(1) = 0. \end{aligned}$$

$$\Rightarrow P_0(x) = \frac{1\cdot(x-1)^0}{0!} = 1.$$

$$P_1(x) = 1 + \frac{2\cdot(x-1)^1}{1!} = 1+2(x-1) = 2x-1$$

$$P_2(x) = (2x-1) + \frac{\cancel{2}\cdot(x-1)^2}{\cancel{2}!} = 2x-1+(x-1)^2$$
$$= \cancel{2x}\cancel{-1}+x^2\cancel{-2x}\cancel{+1} = x^2$$

$$P_3(x) = x^2 + 0\cdot(\text{who cares}) = x^2.$$

Remark: What is the moral of the story?

AEX 21] Let $P_n(x)$ be the n^{th} Taylor Poly. at $x=1$ for $f(x)=\ln(x)$.

1) Show that $\lim_{n\to\infty}(\ln(1.1)-P_n(1.1))=0$.

proof: We shall use the technique explained on page A33.

First, we need to find the derivatives of f and then a sequence of bounds for them — B_n.

As before, $f^{(k)}(x)=(-1)^{k-1}(k-1)!\cdot x^{-k}$

Now, observe that $|f^{(k)}(x)| = |(-1)^{k-1}(k-1)!\,x^{-k}|$

$$=|(k-1)!\,x^{-k}| = \frac{|(k-1)!|}{x^k} \leq |(k-1)!| \text{ for all } x \geq 1.$$

This gives us $B_n=(n-1)!$. Using this, we see that $|\ln(1.1)-P_n(1.1)| = |e_n(1.1)|$

$$\leq B_{n+1}\left|\frac{(1.1-1)^{n+1}}{(n+1)!}\right| = (n-1)!\left|\frac{(1.1-1)^{n+1}}{(n+1)!}\right| = \left|\frac{(n-1)!\,(1.1-1)^{n+1}}{(n+1)!}\right|$$

$$=\left|\frac{(.1)^{n+1}\cancel{(n-1)!}}{(n+1)(n)\cancel{(n-1)!}}\right| = \left|\frac{(.1)^{n+1}}{(n+1)(n)}\right| \underset{\text{for all } n\geq 1}{\leq} (.1)^{n+1} \to 0$$

as $n\to\infty$.

QED

AEX 21] 2) Those of you in tutorial know that there is an ugly way to do this in general. However, for this problem it suffices to see that if we try $n=1$ ... using part 1)

$$|e_1(1.1)| \le \left| \frac{(.1)^{1+1}}{(1+1)(1)} \right| = .005 \quad \text{Too big!}$$

$n=2$ ...

$$|e_2(1.1)| \le \left| \frac{(.1)^{2+1}}{(2+1)(2)} \right| = \left| \frac{(.1)^3}{6} \right| = \frac{.001}{6}$$

$< .001$

So $n=2$ is large enough!

TOPS FORM
7526

Litho in U.S.A.