LECTURE OUTLINE
The Joy of Taylor Series

Professor Leibon

Math 15

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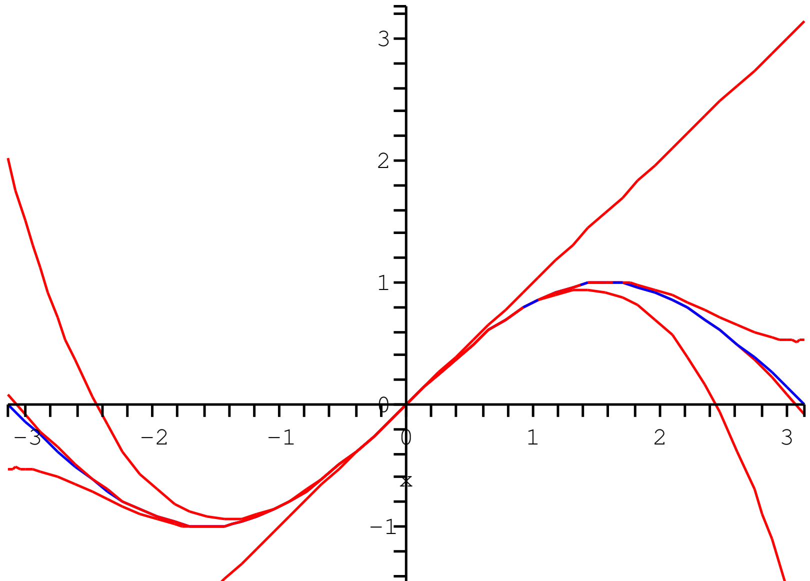
Taylor Approximation: The Remainder

nth Order Approximation at a

$$f(x) \approx \sum_{k=0}^{n} \frac{f^k(a)}{k!}(x-a)^k \equiv P_n(x,a)$$

near a. Notice $\frac{d^k}{dx^k} P_n(x,a)\Big|_{x=a} = f^k(a)$ for all $0 \le k \le n$.

Ex: Find $P_n(x,0)$ for $\sin(x)$.



An Estimate

Theorem: Let $e_n(x) = f(x) - P_n(x, a)$. If for every x in $[a - b, a + b]$ we have that $|f^{n+1}(x)| \leq B_{n+1}$, then

$$|e_n(x)| \leq B_{n+1} \frac{|x - a|^{n+1}}{(n+1)!}.$$

Example: Compute $\sin(1)$ to with in 0.001 (this is $\sin$ of 1 radian).

Controlling the Error

Theorem: Show **on any interval** that $e_n(x)$ tends to zero as n tends to infinity for $\sin(x)$ with $a = 0$.

Hence:

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

Taylor Series

For any x (memorize!)

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

SQUIDOLICIOUS!

Demonstrate (memorize!)

$$e^{ix} = \cos(x) + i\sin(x).$$

Demonstrate (do not memorize!)

$$(\cos(x))^2 - (\sin(x))^2 = \cos(2x)$$

$$2\cos(x)\sin(x) = \sin(2x)$$

Euler's Epitaph

$$e^{i\pi} + 1 = 0$$

Using Taylor Series

Demonstrate

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$\int \sin(x)dx = -\cos(x) + C$$

Very Very Very Very Very Very Very Very Very Very Very Very Very Very Very Very Very Very Very Very Very Very Very Important Example

The following identity **really** wants to hold (memorize!)

$$\frac{1}{1-x} = \sum_{k=1}^{\infty} x^k.$$

Where is this true? Why?