LECTURE OUTLINE
Work and Line Integrals

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Math 8

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Goals

Introduce:

Work

The Line Integral

Derivatives and the Dot Product

Theorem:

$$\frac{d}{dt}(\vec{w_1} \cdot \vec{w_2}) = \frac{d\vec{w_1}}{dt} \cdot \vec{w_2} + \vec{w_1} \cdot \frac{d\vec{w_2}}{dt}$$

Recall, some notation from last time
$\vec{w_1} = x_1(t)\hat{i} + y_1(t)\hat{j} + z_1(t)\hat{k}$ and
$\vec{w_2} = x_2(t)\hat{i} + y_2(t)\hat{j} + z_2(t)\hat{k}$.

One Consequence (of many)

Theorem: If the curvature is not 0, then

$$\hat{N} \cdot \hat{T} = 0.$$

Forces

We must distinguish between the total force

$$\vec{F}_T = m\vec{a}$$

acting on an object and a *force* $\vec{F}_i$ acting on an object where

$$\vec{F}_T = \sum_{i=1}^{n} \vec{F}_i.$$

Forces

Suppose a particle is acted on by a force

$$\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$$

while following the path (t^2, t^3, t) for t in $[0, 1]$. Can $\vec{F}$ be the only force acting on the particle?

Work

Let $\vec{F}$ be a force and let γ denote the the path determined by $\vec{r}(t)$ for t in the interval $[a, b]$. We say the work done by $\vec{F}$ as an object traverses γ is given by the following *line integral*

$$W_{\vec{F}}(\gamma) = \int_{\gamma} \vec{F} \cdot d\vec{r} = \int_{a}^{b} \vec{F} \cdot \frac{d\vec{r}}{dt} dt.$$

(From the first integral $W_{\vec{F}}(\gamma)$ is independent of the parameterization. Yet, from second integral we see work is most easily computed from a given parameterization.)

Example 1(a)

Let

$$\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$$

and γ denote the the path determined by

$$\vec{r}(t) = t^2\hat{i} + t^3\hat{j} + t\hat{k}$$

for t in the interval $[0, 1]$. Compute the work done by $\vec{F}$ as our object traverses γ.

Potential Energy

For each component force, we define the *potential energy* associated to this force at each time t to be

$$U_i(\gamma(t)) = -W_{\vec{F}_i}(\gamma([a,t])).$$

(This notation is sly. It suggests that potential energy should depend only on γ's end points. This is **not** always true, though this is indeed often the case for potential energies of interest to us.)

Example 1(b)

Let

$$\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$$

and let γ denote the the path determined by

$$\vec{r}(t) = t^2\hat{i} + t^3\hat{j} + t\hat{k}$$

for t in the interval $[0, 1]$. Compute the potential energy at $(1, 1, 1)$.

Example 1(c)

Let

$$\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$$

and let γ denote your favorite path determined from $(0, 0, 0)$ to $(1, 1, 1)$. Compute the potential energy at $(1, 1, 1)$.