Math 15

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162 $\vec{u} = \frac{\partial}{\partial x}(x, y, f(x,y))\Big|_{(x_0,y_0)} = (1, 0, \frac{\partial f}{\partial x}(x_0,y_0))$

$\vec{v} = \frac{\partial}{\partial y}(x, y, f(x,y))\Big|_{(x_0,y_0)} = (0, 1, \frac{\partial f}{\partial y}(x_0,y_0))$

are tangent to the graph of $f(x,y)$ @ (x_0,y_0)

$\Rightarrow \vec{n} = \vec{u} \times \vec{v}$ is normal to the tangent plane

$$\vec{n} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x}(x_0,y_0)\,\hat{x} - \frac{\partial f}{\partial y}(x_0,y_0)\,\hat{y} + \hat{z}$$

We know the point $(x_0, y_0, f(x_0,y_0))$ is on the tangent plane

$\Rightarrow (x - x_0, y - y_0, z - f(x_0,y_0)) \cdot \vec{n} = 0$ for any (x,y,z) on the tangent plane

$$\Rightarrow -\frac{\partial f}{\partial x}(x_0,y_0)(x - x_0) - \frac{\partial f}{\partial y}(x_0,y_0)(y - y_0) + (z - f(x_0,y_0)) = 0$$

$$\Rightarrow z = \frac{\partial f}{\partial x}(x_0,y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y - y_0) + f(x_0,y_0)$$

163 $w = \frac{\partial f}{\partial x}(x_0,y_0,z_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0,y_0,z_0)(y - y_0) + \frac{\partial f}{\partial z}(x_0,y_0,z_0)(z - z_0) + f(x_0,y_0,z_0)$

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1. $(x_0, y_0, f(x_0,y_0)) = (2, 1, 5)$

$\frac{\partial f}{\partial x} = 2x \Rightarrow \frac{\partial f}{\partial x}(2,1) = 4$ & $\frac{\partial f}{\partial y} = 2y \Rightarrow \frac{\partial f}{\partial y}(2,1) = 2$

$\Rightarrow z = 4(x-2) + 2(y-1) + 5 = 4x + 2y - 5$

Omit 2. & 3.

165 For the point $(3\sqrt{2}, \frac{3}{2}, 1)$:

$z = \pm\sqrt{36 - x^2 - y^2}$ since $z = 1 > 0$ take $f(x,y) = \sqrt{36 - x^2 - y^2}$

$f(3\sqrt{2}, \frac{3}{2}) = \sqrt{36 - (3\sqrt{2})^2 - (\frac{3}{2})^2} = \frac{3}{2}\sqrt{6}$

$\frac{\partial f}{\partial x} = \frac{1}{2}\left(36 - x^2 - y^2\right)^{-\frac{1}{2}}(-2x) = -x\left(36 - x^2 - y^2\right)^{-\frac{1}{2}}$

$\frac{\partial f}{\partial x}(3\sqrt{2}, \frac{3}{2}) = \frac{-3\sqrt{2}}{\frac{3}{2}\sqrt{6}} = -\frac{2}{\sqrt{3}}$

$\frac{\partial f}{\partial y} = -y\left(36 - x^2 - y^2\right)^{-\frac{1}{2}} \Rightarrow \frac{\partial f}{\partial y}(3\sqrt{2}, \frac{3}{2}) = \frac{-\frac{3}{2}}{\frac{3}{2}\sqrt{6}} = -\frac{1}{\sqrt{6}}$

Tangent plane: $z = -\frac{2}{\sqrt{3}}(x - 3\sqrt{2}) - \frac{1}{\sqrt{6}}(y - \frac{3}{2}) + \frac{3}{2}\sqrt{6}$

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1. $f(x,y) = \sqrt{x^2 + y^2} \Rightarrow \frac{\partial f}{\partial x} = x(x^2 + y^2)^{-\frac{1}{2}}$ $\frac{\partial f}{\partial y} = y(x^2 + y^2)^{-\frac{1}{2}}$

$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$

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1. $\frac{\partial f}{\partial \vec{u}} = \nabla f \cdot \vec{u}$ @ $(2,1)$

$\nabla f(2,1) = \left(\frac{2}{\sqrt{4+1}}, \frac{1}{\sqrt{4+1}}\right) = \left(\frac{2}{5}, \frac{1}{5}\right)$

$\frac{\partial f}{\partial \vec{u}}(2,1) = \nabla f(2,1) \cdot \left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{2}{5}, \frac{1}{5}\right) \cdot \left(\frac{3}{5}, \frac{4}{5}\right) = \frac{2}{5}$

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1. $\frac{\partial f}{\partial \vec{u}} = \nabla f \cdot \vec{u} = |\nabla f||\vec{u}| \cos\theta$ is greatest when $\theta = 0$

$\Rightarrow$ direction of ∇f & $\vec{u}$ are the same

2. $\frac{\partial f}{\partial \vec{u}} = |\nabla f| \, |\hat{u}| \cos\theta$ but $|\hat{u}| = 1$

$\Rightarrow \frac{\partial f}{\partial \vec{u}} = |\nabla f|$ @ $\theta = 0$

3. $\nabla f \cdot \vec{u} = 0$ iff $\nabla f \perp \vec{u}$

4. $\frac{\partial f}{\partial(-\vec{u})} = |\nabla f| \cos\theta = -|\nabla f| \cos(\theta + \pi) = -\frac{\partial f}{\partial \vec{u}}$

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$\nabla T = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right) = (y, x, -2z)$

$\nabla T(3,4,5) = (4, 3, -10)$

$\Rightarrow \vec{u} = \frac{1}{\sqrt{4^2 + 3^2 (-10)^2}} (4, 3, -10) = \left(\frac{4}{5\sqrt{5}}, \frac{3}{5\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)$

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$g(x,y,z) = f(x,y) - z$

1. $\nabla g = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\right)$

2. ∇g is normal to level sets of g

$g(x,y,z) = 0 = f(x,y) - z \implies z = f(x,y)$

$\implies \nabla g(x_0, y_0, f(x_0, y_0))$ is normal to the graph of $f(x,y)$

3. $\nabla g(x_0, y_0, f(x_0, y_0)) \cdot (x - x_0, y - y_0, z - f(x_0, y_0)) = 0$

$\implies \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) - z + f(x_0, y_0) = 0$