LECTURE OUTLINE

Tangent Planes and Gradients

Professor Leibon

Math 15

Oct. 27, 2004

Goals

The Chain Rule

The Gradient

The Tangent Plane

Chain Rule

Recall

$$\frac{df(x)}{dt} = \frac{df}{dx}\frac{dx}{dt},$$

this chain rule generalizes, and we have

$$\frac{df(x,y)}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Ex. Let $f(x,y) = x^2y + y^3$, $x(t) = \sin(t)$, and $y(t) = e^t$. Find $\frac{d}{dt}\left(e^t(\sin(t))^2 + e^{3t}\right)$ in the old way and using the chain rule.

Contour Directions and Mountain Climbs

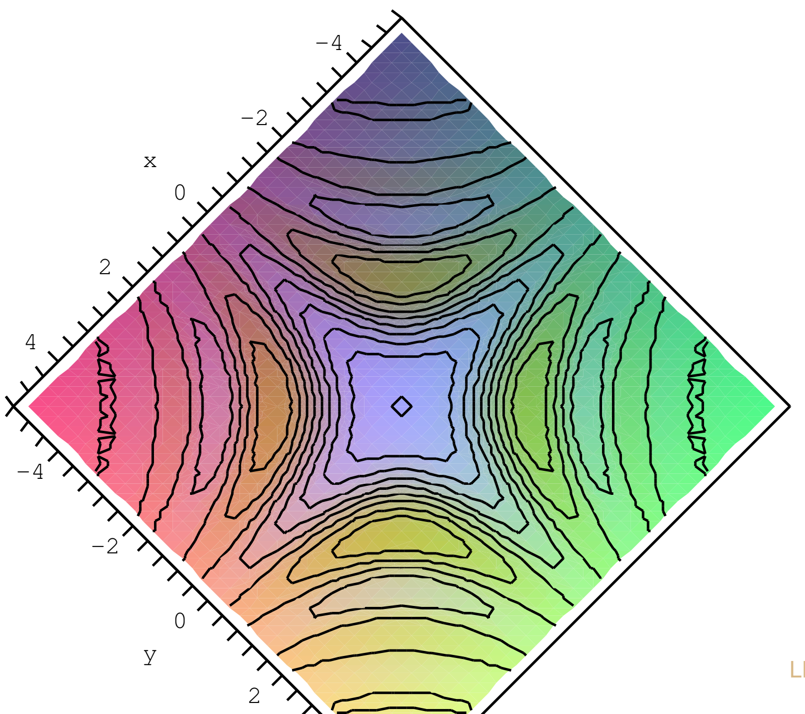
Pick a direction $\hat{u} = \cos(\theta)\hat{i} + \sin(\theta)\hat{j}$ in the contour plot at the point (x_0, y_0). Travel in this direction via $(x_0 + t\cos(\theta))\hat{i} + (y_0 + t\sin(\theta))\hat{j}$ in the contour map. What is our velocity vector on the mountain side at $(x_0, y_0, f(x_0, y_0))$?

Answer: Letting $\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$, we have our velocity vector on the mountain side is given by

$$\vec{u} + (\nabla f \cdot \vec{u})\hat{k}.$$

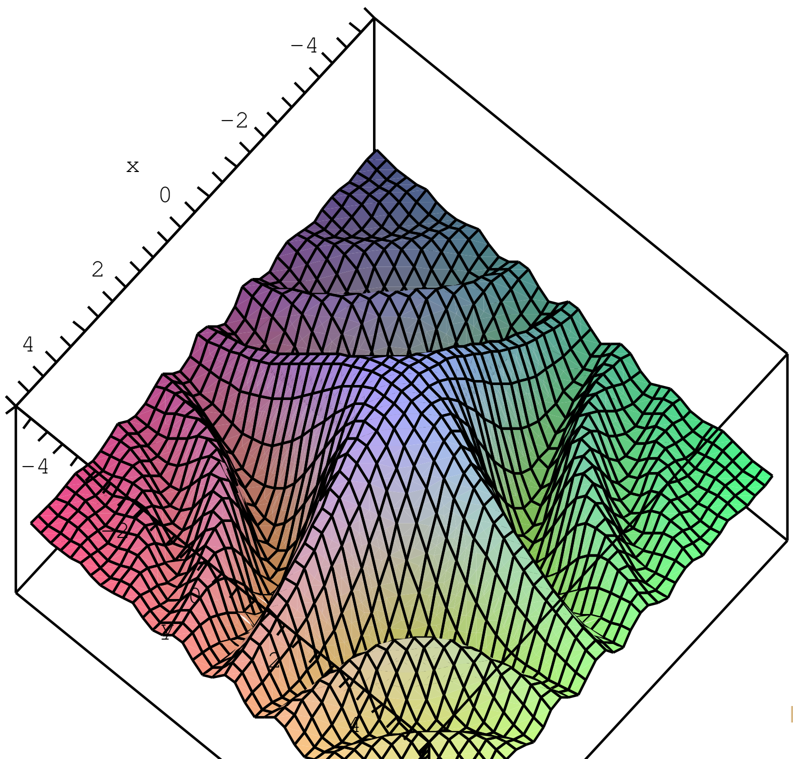
Example

Let $f(x,y) = \cos(xy)e^{\frac{-x^2-y^2}{10}}$. Find the gradient of $f(x,y)$. In each contour direction, what is our velocity vector at $(1, 0, e^{-\frac{1}{10}})$. What is the rate of change of our height in each of these directions?



Interpreting the Gradient

Notice: the fastest rate of height gain is achieved in the direction $\frac{\nabla f}{|\nabla f|}$, the fastest rate of height loss is achieved in the direction $-\frac{\nabla f}{|\nabla f|}$, while there is no change in height in a direction perpendicular to $\frac{\nabla f}{|\nabla f|}$.



Plane

A plane thought the origin is the collection of all position vectors $\vec{r}$ perpendicular to a fixed vector $\vec{n}$, the plane's *normal vector*. Hence the equation $\vec{n} \cdot \vec{r} = 0$ determines such a plane.

While the plane parallel to this plane that contains the point $\vec{p}$ is determined by the equation $\vec{n} \cdot (\vec{r} - \vec{p}) = 0$.

Notice, a plane containing the vectors $\vec{v}_1$ and s $\vec{v}_2$ has $\vec{n} = \vec{v}_1 \times \vec{v}_2$.

Tangent Plane

The tangent plane at a point on a mountain side is the plane contain the velocity vectors of the curves going through that point.

Argue that: the tangent plane at $(x_0, y_0, f(x_0, y_0))$ has normal given by $-\frac{\partial f}{\partial y}\hat{i} - \frac{\partial f}{\partial x}\hat{j} + \hat{k}$.

Find an equation for the tangent plane of $f(x, y) = \cos(xy)e^{\frac{-x^2-y^2}{10}}$ at $(1, 0, e^{-\frac{1}{10}})$.