LECTURE OUTLINE
Kinematics of Rotation

Professor Leibon

Math 15

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Goals

Cross Product and the Determinant

Rigid Body Kinematics

Last time we explored the Cross Product

Nice and Linear:

$$(\vec{v}_1 + c\vec{v}_2) \times \vec{w} = \vec{v}_1 \times \vec{w} + c\vec{v}_2 \times \vec{w}$$

but, Not Commutative!

$$\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$$

and Not Associative!

$$(\vec{u} \times \vec{v}) \times \vec{w} = \vec{u} \times (\vec{v} \times \vec{w}) + \vec{v} \times (\vec{w} \times \vec{u})$$

In particular, $\vec{u} \times \vec{v} \times \vec{w}$ has no good meaning!

Computing a 3×3 Determinant

$$\vec{u} \times \vec{v} = \begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix} \hat{i} - \begin{vmatrix} u_x & u_z \\ v_x & v_z \end{vmatrix} \hat{j} + \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \hat{k}$$

$$\vec{w} \cdot (\vec{u} \times \vec{v}) \equiv \begin{vmatrix} w_x & w_y & w_z \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = \vec{w} \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix}$$

Let $\vec{u} = \hat{i} - 2\hat{k}$, $\vec{v} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{w} = 3\hat{i} + 2\hat{j}$, and compute $\vec{w} \cdot (\vec{u} \times \vec{v})$

Application to spacial reasoning

Ex: Note we can think of a line as $t\vec{v}+\vec{p}$. Define what it means for two lines to be non-parallel, and find a formula for the distance between two non-parallel lines.

Find the distance between the lines $(\hat{i}-2\hat{k})t+\hat{i}$ and $\vec{v}=t(\hat{i}+\hat{j}+\hat{k})$.

Kinematics

For our rotation use $\hat{i}$, $\hat{j}$, and $\hat{k}$ with $\hat{k}$ the axis of rotation and $\hat{i}$ to $\hat{j}$ in the direction of rotation (right hand rule). Give $\hat{r}$ and $\hat{\theta}$ there usual meanings with respect to $\hat{i}$, $\hat{j}$, $\hat{k}$. There is a second "view" we might take to think about the center (center of mass) which we might call $\hat{i}_c$, $\hat{j}_c$, $\hat{k}_c$. A particle moving about another particle can be described by:

$$\vec{r} = \vec{c} + r\hat{r} + z\hat{k}.$$

Kinematics

Usual Book Assumptions:

(1) $\dot{r} = 0$ (Not Nutty)

(2) $z = 0$ (Not so Nutty)

(3) $\frac{d\hat{k}}{dt} = 0$ (Nutty)

Note: (3) **The Great Yo-Yo Restriction** allows us to assume $\frac{d\hat{i}}{dt} = 0$ and $\frac{d\hat{j}}{dt} = 0$ as well.

Kinematics

Book's Notation under the Usual Book Assumptions:

$$\vec{\omega} \equiv \dot{\theta}\hat{k} \equiv \omega\hat{k}$$

$$\vec{\alpha} \equiv \frac{d\vec{\omega}}{dt} = \ddot{\theta}\hat{k}$$

$$\vec{v} \equiv \frac{d\vec{r}}{dt} = r\dot{\theta}\hat{\theta} = \vec{\omega} \times \vec{r}$$

$$\vec{a} \equiv \frac{d\vec{v}}{dt} = -r\dot{\theta}^2\hat{r} + r\ddot{\theta}\hat{\theta} = \vec{\omega} \times \vec{v} + \vec{\alpha} \times \vec{r}$$

Derivatives in Polar Coordinates

$$\frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta}$$

$$\frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{r}$$

$$\frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\frac{d^2\vec{r}}{dt^2} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$