LECTURE OUTLINE
Cross Product

Professor Leibon

Math 15

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Goals

Cross Product

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Given vectors $\vec{u}$ and $\vec{v}$ we define $\vec{u} \times \vec{v}$ to be the unique vector satisfying

(1) $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$ and to $\vec{v}$ (or zero).

(2) It has length equal to the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$.

(3) $\vec{u} \times \vec{v}$ is in the direction determined by the right hand rule going from $\vec{u}$ to $\vec{v}$.

Example: Find $\hat{r} \times \hat{\theta}$.

Main Theorem

Let $u = u_x\hat{i} + u_y\hat{j} + u_z\hat{k}$ and $v = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$, then $\vec{u} \times \vec{v}$ equals

$$(u_yv_z - u_zv_y)\hat{i} - (u_xv_z - u_zv_x)\hat{j} + (u_xv_y - u_yv_x)\hat{k}.$$

Example: Find $\hat{r} \times \hat{\theta}$.

Prove property (1).

Warm Up: Parallelogram Area in the plane

The oriented area A of the parallelogram determined by $\vec{u} = a\hat{i} + b\hat{j}$ and $\vec{v} = c\hat{i} + d\hat{j}$ satisfies

$$A^2 = (base)^2(height)^2 = |\vec{u}|^2|\vec{v}|^2(\sin(\theta))^2$$

$$= (a^2+b^2)(c^2+d^2)(1-\cos(\theta)^2) = ((a^2+b^2)(c^2+d^2)-(ac+bd)^2)$$

$$= (ad-bc)^2$$

Hence the square root, *the oriented area*, is given by

$$ad - bc \equiv \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

(The last line is "absolutely" the **stupidest notation ever introduced.** Why?)

Parallelogram Area (Part (2))

The oriented area A of the parallelogram determined by $\vec{u} = u_x\hat{i} + u_y\hat{j} + u_z\hat{k}$ and $\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$, satisfies

$$A^2 = (base)^2(height)^2 = |\vec{v}|^2(|\vec{w}|^2(1 - \cos(\theta)^2)$$

$$= (u_x^2 + u_y^2 + u_z^2)(v_x^2 + v_y^2 + v_z^2) - (u_xv_x + u_yv_y + u_zv_z)^2$$

$$= (u_yv_z - u_zv_y)^2 + (u_xv_z - u_zv_x)^2 + (u_xv_y - u_yv_x)^2$$

$$= \begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix}^2 + \begin{vmatrix} u_x & u_z \\ v_x & v_z \end{vmatrix}^2 + \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}^2$$

Ex. Find the area of the parallelogram determined by $\vec{u} = \hat{i} - 2\hat{k}$ and $\vec{v} = \hat{i} + \hat{j} + \hat{k}$.

Parallelepiped Volume

The volume V^2 of the parallelepiped determined by $\vec{u}$, $\vec{v}$, and $\vec{w}$ is given by

$$V^2 = (base)^2(height)^2 = |\vec{u} \times \vec{v}|^2 \left(|\vec{w}| \left(\frac{(\vec{u} \times \vec{v}) \cdot \vec{w}}{|\vec{u} \times \vec{v}||\vec{w}|} \right) \right)^2$$

$$= (\vec{w} \cdot (\vec{u} \times \vec{v}))^2$$

Hence the square root, *the oriented volume*, is given by

$$\vec{w} \cdot (\vec{u} \times \vec{v}) \equiv \det \begin{pmatrix} w_x & w_y & w_z \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{pmatrix} \equiv \begin{vmatrix} w_x & w_y & w_z \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix}.$$

Parallelepiped Volume

Ex. Find the area of the parallelepiped determined by $\vec{u} = \hat{i} - 2\hat{k}$, $\vec{v} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{w} = 3\hat{i} + 2\hat{j}$.

Ex: Note we can think of a line as $t\vec{v} + \vec{p}$. Define what it means for two lines to be non-parallel, and find a formula for the distance between two non-parallel lines.