LECTURE OUTLINE

Fun With Power Series!

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Math 15

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Goals

Review Power Series

Let's Party! (Sum fun sums)

Power Series

A function given by

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n.$$

is called a power series. It domain is the set of x where this series converges.

Ex: $\sum_{n=1}^{\infty} x^n$, $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$, $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$, and $\sum_{k=0}^{\infty} \frac{x^k}{k!}$.

Radius of Convergence

$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ either

(a) converges only at a

(b) converges for all x

(c) there is an $R > 0$ called the *radius of convergence* such that $f(x)$ converges or all x such that $a - R < x < a + R$ and diverges for all $a + R < x$ and $x < a - R$.

Ex: Recall the radius of convergence of $\sum_{n=1}^{\infty} x^n$, $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$, $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$, and $\sum_{k=0}^{\infty} \frac{x^k}{k!}$.

Differentiation and Integration

If $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ has a radius of convergence R, then

$$\frac{df}{dx} = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}, \text{ and}$$

$$\int f\,dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1} + C$$

in $(a-R, a+R)$, and each of these power series has radius of convergence R.

Examples

Ex: Find a power series expansion of $\log(1-x)$ about $x = 0$ and its radius of convergence.

Ex: Find a power series expansion of $\arctan(x)$ about $x = 0$ and find its radius of convergence.

Euler's γ Constant

Show the *harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverge to "$\ln(\infty)$". We Let

$$\lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{n} - \ln(N) \right) = \gamma \approx 0.57721566$$

Show

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln(2).$$

Verify

$$
\begin{array}{ccccccccccccccccccc}
1 & - & \frac{1}{2} & + & \frac{1}{3} & - & \frac{1}{4} & + & \frac{1}{5} & - & \frac{1}{6} & + & \frac{1}{7} & - & \frac{1}{8} & + & \frac{1}{9} & \ldots \\
+ \\
0 & + & \frac{1}{2} & + & 0 & - & \frac{1}{4} & + & 0 & + & \frac{1}{6} & + & 0 & - & \frac{1}{8} & + & 0 & \ldots \\
= \\
1 & + & 0 & + & \frac{1}{3} & - & \frac{1}{2} & + & \frac{1}{5} & + & 0 & + & \frac{1}{7} & - & \frac{1}{4} & + & \frac{1}{9} & \ldots
\end{array}
$$

Does this imply $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \frac{3}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ equals $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$?

In Between Alternating and Positive

Recall, we can factor a number into *distinct primes* raised to some *powers*. For example,

$$12 = (2)^2(3)$$

while

$$30 = (2)(3)(5).$$

We say that 30 has 3 *distinct factors* (2,3, and 5), while we say that 12 has a factor with a *power* bigger than 1 (namely the 2 is squared).

In Between

Let $f(n)$ be equal to the $Möbius$ *Function* which is

0 if some factor of n has a *power* bigger than 1,

$+1$ if the factors are distinct and there are an even number of them,

and -1 if the factors are distinct and there are an odd number of them.

The Challenge: Find where $\sum \frac{Möbius(n)}{n^p}$ converges.

Thinking about the Radius of Convergence

For any a, find b_n so that near a

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} b_n (x-a)^k$$

and find this series's radius of convergence.

Exciting Example: Try and do the same for $\frac{1}{1+x^2}$.

Exciting Examples

Explore the radius of convergence of the power Series expansion of

$$\frac{1}{1+x^2}$$

about $a = 0$, $a = 1$, and $a = 2$.