Exam 2 **Name:**

You May Use:

$$\frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta}$$

$$\frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{r}$$

$$\frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\frac{d^2\vec{r}}{dt^2} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$

$$|f(x) - P_n(x,a)| \leq B_{n+1}\frac{|x-a|^{n+1}}{(n+1)!}.$$

Under the books assumptions: $\dot{r} = 0$, $z = 0$, and $\frac{d\hat{k}}{dt} = 0$:

$$\vec{\omega} \equiv \dot{\theta}\hat{k} \equiv \omega\hat{k}$$

$$\vec{\alpha} \equiv \frac{d\vec{\omega}}{dt} = \ddot{\theta}\hat{k}$$

$$\vec{v} \equiv \frac{d\vec{r}}{dt} = r\dot{\theta}\hat{\theta} = \vec{\omega} \times \vec{r}$$

$$\vec{a} \equiv \frac{d\vec{v}}{dt} = -r\dot{\theta}^2\hat{r} + r\ddot{\theta}\hat{\theta} = \vec{\omega} \times \vec{v} + \vec{\alpha} \times \vec{r}$$

Solutions

1. Find the real and imaginary parts of $\frac{1}{2z+3}$ with $z = x + iy$.

$$w = \frac{1}{2z+3} = \frac{1}{2(x+iy)+3}$$

$$= \frac{1}{(2x+3)+i(2y)} = \frac{1}{(2x+3)+i(2y)} \cdot \frac{(2x+3)-i(2y)}{(2x+3)-i2y}$$

$$= \frac{(2x+3)-i(2y)}{(2x+3)^2+(2y)^2}$$

So

$$\boxed{\mathrm{Re}(w) = \frac{2x+3}{(2x+3)^2+4y^2}}$$

&

$$\boxed{\mathrm{Im}(w) = \frac{-2y}{(2x+3)^2+4y^2}}$$

2. Find two distinct vectors of length 1 which are simultaneously perpendicular to line through $(1,2,1)$ and $(2,-2,0)$ and the line through $(2,1,3)$ and $(-1,4,-2)$.

We the lines share their directions with $\vec{v} = (1,2,1) - (2,-2,0) = (-1,4,1)$

& $\vec{w} = (2,1,3) - (-1,4,-2) = (3,-3,5)$

hence are both perpendicular to

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 4 & 1 \\ 3 & -3 & 5 \end{vmatrix} = 23\hat{i} + 8\hat{j} - 9\hat{k}.$$

Letting $C = |\vec{v} \times \vec{w}| = \sqrt{23^2 + 8^2 + 9^2}$, our two vectors are

$$\boxed{\vec{u}_1 = \frac{23}{C}\hat{i} + \frac{8}{C}\hat{j} - \frac{9}{C}\hat{k}}$$

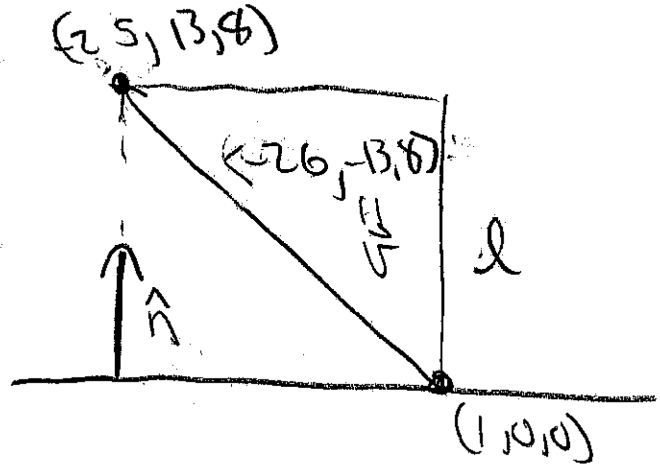
&

$$\boxed{\vec{u}_2 = -\vec{u}_1}$$

3. Find the distance between the point $P = (-25, -13, 8)$ and the plane with equation $3x + y - z = 3$.

(See Example 54 in book)
(Method 2 observes that...)

we need ℓ in



since
$3 \cdot 1 + 0 - 0 = 3$

where

$\hat{n} = \dfrac{(3,1,-1)}{\sqrt{3^2+1+1}}$ since $\vec{n} \cdot \vec{p} = c$ is our plane with $\vec{n}$ it's normal.

so

$$\ell = \left|\vec{v} \cdot \hat{n}\right| = \left|\frac{\langle -26,-13,8\rangle \cdot \langle 25,13,8\rangle}{\sqrt{11}}\right|$$

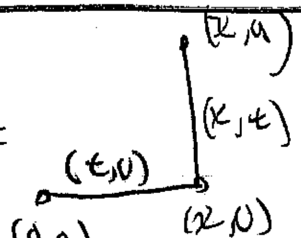
$$= \frac{99}{\sqrt{11}} = \boxed{9\sqrt{11}}$$

4. Determine whether each of the following force fields are conservative. If it is, then find a potential function. Justify your answers.

(a) $\vec{F}(x, y, z) = (y\sin(y^2), y)$.

Method 1

If V exist, then $V = -\int_\gamma \vec{F}\cdot d\vec{r}$ with $\gamma =$



$$= -\int_0^x (0,0)\cdot(1,0)\,dt - \int_0^y (t\sin(t^2), t)\cdot(0,1)\,dt$$

$$= -\int_0^y t\,dt = -\frac{y^2}{2}.$$ But $-\frac{\partial}{\partial x}\left(-\frac{y^2}{2}\right) = 0 \neq y(\sin(y^2))$

So $\boxed{\text{V does not exist.}}$

Method 2

If V exists, then $-\frac{\partial Q}{\partial x} = y\sin(y^2)$ & $-\frac{\partial Q}{\partial y} = y$

$-Q = xy(\sin(y^2)) + f(y)$ $\quad -Q = \frac{y^2}{2} + g(x)$

Well from this

$-\frac{\partial Q}{\partial x} = y\sin(y^2) = \frac{dg}{dx}(x)$. A contradiction

since $\frac{dg}{dx}(x)$ is a function of x.

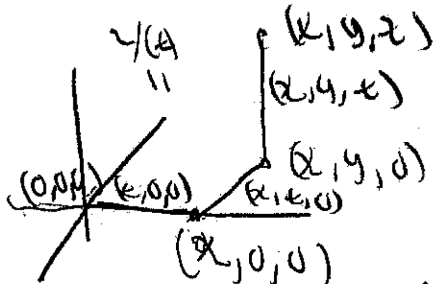
So $\boxed{\text{V does not exist}}$

Method 1

(b) $\vec{F}(x, y) = (yz, xz + y^2, xy)$.

If V exist, then $V = -\int_\gamma \vec{F} \cdot d\vec{r}$

with



(this path γ has 3 pieces)

$$V = -\int_0^x (0,0,0)\cdot(1,0,0)\,dt - \int_0^y (0,t^2,xt)\cdot(0,1,0)\,dt - \int_0^z (yt, xt+y^2, xy)\cdot(0,0,1)\,dt$$

$$= -0 - \int_0^y t^2\,dt - \int_0^z xy\,dt = -\frac{y^3}{3} - xyz \quad \&$$

& $-\nabla V \overset{\partial}{=} (xy, xz+y^2, xy)$ hence $\boxed{V = -\frac{y^3}{3} - xyz}$

Method 2

If V exists $\frac{\partial a}{\partial x} = -yz$ $\quad \frac{\partial a}{\partial y} = -xz - y^2$ $\quad \frac{\partial a}{\partial z} = -xy$

$a = -yzx + f_1(y,z)$ $\quad a = -xzy - \frac{y^3}{3} + f_2(x,z)$ $\quad a = -xyz + f_3(x,y)$

& note setting $f_1(y,z) = f_3(x,y) = -\frac{y^3}{3}$ & $f_2(x,z) = 0$

we have

$a = \boxed{-xyz - \frac{y^3}{3} = V}$ our needed potential.

5. (a) What is the Taylor series of $\sin(x)$ around the point $x = 0$?

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

(b) What is the Taylor series of $\cos(x)$ around the point $x = 0$?

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

(c) What is the Taylor series of e^x around the point $x = 0$?

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Squidolicious!!!

(d) Using the series from (a)-(c), justify that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

$$e^{i\theta} \underset{\text{by (c)}}{=} \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} = \sum_{\text{even } k} \frac{(i\theta)^k}{k!} + \sum_{\text{odd } k} \frac{(i\theta)^k}{k!}$$

$$= \sum_{n=0}^{\infty} \frac{(i\theta)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(i\theta)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(i^2)^n \theta^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(i^2)^n i^1 \theta^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} + i\sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!}$$

$$\underset{\text{by (a) \& (b)}}{=} \cos(\theta) + i\sin(\theta)$$

as needed.

6. Let $\vec{u}(3) = (1,-1,2)$, $\vec{v}(3) = (3,0,-1)$, $\frac{d\vec{u}}{dt}(3) = (1,2,0)$, $\frac{d\vec{v}}{dt}(3) = (0,-1,2)$ and $\nabla f(1,-1,2) = (2,5,3)$.

(a) Compute $\frac{d}{dt}(f(\vec{u}(t))$ at $t=3$.

$$\left.\frac{d}{dt} f(\vec{u}(t))\right|_{t=3} \overset{\text{chain Rule}}{=} \left.\nabla f(\vec{u}(t)) \cdot \frac{d\vec{u}}{dt}(t)\right|_{t=3}$$

$$= (2,5,3)\cdot(1,2,0) = \boxed{12}$$

(b) Compute $\frac{d}{dt}(\vec{u}\cdot\vec{v})$ at $t=3$.

$$\left.\frac{d}{dt}(\vec{u}\cdot\vec{v})\right|_{t=3} = \left.\frac{d\vec{u}}{dt}\cdot\vec{v} + \vec{u}\cdot\frac{d\vec{v}}{dt}\right|_{t=3}$$

$$= (1,2,0)\cdot(3,0,-1) + (1,-1,2)\cdot(0,-1,2)$$

$$= 3 + 5 = \boxed{8}$$

(c) Compute $\frac{d}{dt}(\vec{u} \times \vec{v})$ at $t = 3$.

$$\frac{d}{dt}(\vec{u} \times \vec{v})\Big|_{t=3} = \frac{d\vec{u}}{dt} \times \vec{v} + \vec{u} \times \frac{d\vec{v}}{dt}\Big|_{t=3}$$

$$= (1,2,0) \times (3,0,-1) + (1,-1,2) \times (0,-1,2)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 0 \\ 3 & 0 & -1 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ 0 & -1 & 2 \end{vmatrix}$$

$$= (-2, 1, -6) + (0, -2, -1)$$

$$= \boxed{(-2, -1, -7)}$$

7. A particle travels around a circle with constant speed $7\frac{meters}{sec}$ starting at $2\hat{i}$ *meters* . (In other words, $\hat{\theta}\cdot\frac{d}{dt}\vec{r}=7$). The radius of the circle changes with time as $r(t)=\frac{2}{1+t}meters$ with $t\geq 0$.

(a) Express $\vec{r}(t)$ in Polar coordinates.

$$7=\hat{\theta}\cdot\frac{dr}{dt}=\hat{\theta}\cdot(\dot{r}\hat{r}+r\dot{\theta}\hat{\theta})=r\dot{\theta}=\frac{2}{1+t}\dot{\theta}$$

so $\dot{\theta}=\left(\frac{1+t}{2}\right)7=\frac{7}{2}(1+t)$ $\quad\theta(t)=\frac{7}{2}\left(t+\frac{t^2}{2}\right)+C$

$\theta(0)=0$ since we start at $2\hat{i}$, so $C=0$

& $\boxed{\vec{r}(t)=r(t)\,\hat{r}(\theta(t))=\frac{2}{1+t}\hat{r}\left(\frac{7}{2}\left(t+\frac{t^2}{2}\right)\right)}$

which $=\left(\frac{2}{1+t},\frac{7}{2}\left(t+\frac{t^2}{2}\right)\right)_p$

(b) Express $\vec{r}(t)$ in Cartesian coordinates.

well

$$\vec{r}(t)=r(t)\,\hat{r}(\theta(t))=r(t)\left(\cos(\theta(t))\hat{i}+\sin(\theta(t))\hat{j}\right)$$

$$=\boxed{\frac{2}{1+t}\cos\left(\frac{7}{2}\left(t+\frac{t^2}{2}\right)\right)\hat{i}+\frac{2}{1+t}\sin\left(\frac{7}{2}\left(t+\frac{t^2}{2}\right)\right)\hat{j}}$$

8. Suppose you have a function $f(x)$ such that $f(x)$'s third Taylor polynomial at $x = 1$ is $P_3(x) = 1 - (1/2)(x-1) + (x-1)^2 + (2/3)(x-1)^3$, and assume that all of $f(x)$'s derivatives are bounded by 5 on the interval $(0, 2)$ (in other words $\left|\frac{d^n f}{dx^n}(x)\right| < 5$ for $0 \le x \le 2$).

   (a) Given the above data, approximate $f(1.5)$.

$$f(1.5) = f\left(\tfrac{3}{2}\right) = 1 - \tfrac{1}{2}\cdot\tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 + \tfrac{2}{3}\left(\tfrac{1}{2}\right)^3$$
$$= 1 + \tfrac{1}{12} = \boxed{\tfrac{13}{12}}$$

   (b) Bound the difference $|f(1.5) - P_3(1.5)|$ using the above data, and justify your answer.

well $\left|\left|f\left(\tfrac{3}{2}\right) - P_3\left(\tfrac{3}{2}\right)\right|\right| \le \dfrac{M\left|\tfrac{3}{2}-1\right|^4}{4!} \le \boxed{\dfrac{5}{2^7 \cdot 3}} \le \dfrac{6}{2^7 \cdot 3} = \dfrac{1}{64}$

& $\left|\underset{\text{over } [1,\frac{3}{2}]}{\max}|f^4(x)|\right| \le \left|\underset{\text{over } [0,2]}{\max}|f^4(x)|\right| = 5$ so

   (c) Given the above data, can you determine $f(x)$'s second derivative at $x = 1$? If so find it, if not why.

Well $P_3(x) = f(1) + f^1(1)\cdot(x-1) + \dfrac{f^2(1)}{2!}(x-1)^2 + \dfrac{f^3(1)}{3!}(x-1)^3$

& so $\dfrac{f^2(1)}{2!} = 1$ & $\boxed{f^2(1) = 2}$