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1. Express the following in $a+bi$ form:

(a) $(2+3i)+(4+i) = 6+4i$

(b) $(2+3i)/(4+i) = \left(\frac{2+3i}{4+i}\right)\left(\frac{4-i}{4-i}\right) = \frac{8-2i+12i+3}{16+1}$

$= \frac{5}{17} + i\frac{10}{17}$

(c) $1/i + 3/(1+i) = -i + \frac{3(1-i)}{1^2+1^2} = -i + \frac{3}{2} - \frac{3}{2}i$

$= \frac{3}{2} - \frac{5}{2}i$

2.

(a) $(2+3i)(4+i) = 8+2i+12i-3 = 5+14i$

(b) $(8+6i)^2 = 64+96i-36 = 28+96i$

(c) $\left(1+\frac{3}{1+i}\right)^2 = \left(1+\frac{3}{2}-\frac{3}{2}i\right)^2 = \left(\frac{5}{2}-\frac{3}{2}i\right)^2 =$

$\frac{25}{4} - \frac{30}{4}i - \frac{9}{4} = \frac{16}{4} - \frac{15}{2}i = 4 - \frac{15}{2}i$

4. (a) $(z+1)^2 = 3+4i = 5\left(\frac{3}{5}+\frac{4}{5}i\right) = 5\cdot e^{i\theta} = 5\cdot e^{i(\theta+2\pi k)}$

$\Rightarrow z+1 = \sqrt{5}\cdot e^{\frac{i(\theta+2\pi k)}{2}} = \sqrt{5}\cdot e^{i\theta/2}\cdot e^{i\pi k} = \pm\sqrt{5}e^{i\theta/2}$

$\Rightarrow z = -1 \pm \sqrt{5}\, e^{i\,\mathrm{atan}(\frac{4}{3})/2}$.

4. (b) $z^4 - i = 0 \iff z^4 = i = e^{i\pi/2} = e^{i(\frac{\pi}{2} + 2\pi k)}$

$\Rightarrow z = e^{i(\frac{\pi}{2}+2\pi k)/4} = e^{i\frac{\pi}{8}} \cdot e^{i\frac{\pi}{2}k}$

So z can be one of: $\begin{cases} \cos(\frac{\pi}{8}) + i\sin(\frac{\pi}{8}) \\ \cos(\frac{5\pi}{8}) + i\sin(\frac{5\pi}{8}) \\ \cos(\frac{9\pi}{8}) + i\sin(\frac{9\pi}{8}) \\ \cos(\frac{13\pi}{8}) + i\sin(\frac{13\pi}{8}) \end{cases}$

5.

(a) $1/z^2 = \frac{1}{(x+iy)^2} = \frac{1}{x^2+2ixy+y^2} = \frac{1}{(x^2-y^2)+2ixy}$

$= \frac{(x^2-y^2)-2ixy}{(x^2-y^2)^2+4x^2y^2} = \frac{(x^2-y^2)}{(x^2-y^2)+4x^2y^2} - i\frac{2xy}{(x^2-y^2)+4x^2y^2}$

(b) $\frac{1}{3z+2}$ = $\frac{1}{3x+3iy+2} = \frac{(3x+2)-3yi}{(3x+2)^2+(9y^2)}$

$= \frac{3x+2}{(3x+2)^2+9y^2} - i\frac{3y}{(3x+2)^2+9y^2}$

17. (a) $(1+i)^4 = \left(\sqrt{2}\cdot e^{i\frac{\pi}{4}}\right)^4 = 4\cdot e^{i\pi} = -4$

(b) $(-i)^{-1} = \frac{(-1)}{i} = \left(\frac{-1}{i}\right)\frac{i}{i} = \frac{-i}{-1} = i$

18. (a) $(1-i)^{-1} = \frac{1}{1-i} = \frac{1+i}{1^2+1^2} = \frac{1}{2} + \frac{1}{2}i$

(b) $\frac{(1+i)}{(1-i)} = \frac{(1+i)^2}{1^2+1^2} = \frac{1+2i-1}{2} = i$

3. Let $z = \frac{(3+8i)^4}{(1+i)^{10}}$. Then $z = \frac{\left(\sqrt{73}\cdot e^{i\theta_1}\right)^4}{\left(\sqrt{2}\cdot e^{i\pi/4}\right)^{10}}$

$$= \frac{73^2 \cdot e^{i4\theta_1}}{32 \cdot e^{i\frac{5\pi}{2}}} = \frac{73^2}{32} \cdot e^{i(4\theta_1 - 5\frac{\pi}{2})}$$

Thus $\bar{z} = \frac{73^2}{32} \cdot e^{i(5\frac{\pi}{2} - 4\theta_1)}$, whcn $\theta_1 = \operatorname{atan}\left(\frac{8}{3}\right)$.

7. $\left|\frac{i(2+3i)(5-2i)}{-2-i}\right| = \left|\frac{(2+3i)(5-2i)}{2+i}\right| = \left|\frac{10-4i+15i+6}{2+i}\right|$

$$= \left|\frac{16+11i}{2+i}\right| = \left|\frac{(16+11i)(2-i)}{2^2+1^2}\right| = \frac{|32-16i+22i+11|}{5} = \frac{|43+6i|}{5}$$

$$= \frac{\sqrt{43^2+36}}{5}$$

15. No. Counterexample: $z = i$.

But when is it true?

Suppose $z^2 = |z^2|$.

$\Rightarrow (re^{i\theta})^2 = |(re^{i\theta})^2|$

$\Rightarrow r^2 e^{i2\theta} = |r^2 e^{i2\theta}| = r^2$

$\Rightarrow r^2 e^{i2\theta} = r^2 \Rightarrow e^{i2\theta} = 1 \Rightarrow 2\theta = 0 + 2\pi K$

$\Rightarrow \theta = \pi K$. In other words, when z is real.

19. We need an equation for which $d(z, 8+5i) = 3$ for all z that satisfy it...

how about $|z - (8+5i)| = 3$?