LECTURE OUTLINE
Functions of Complex Numbers

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Math 15

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Goal

Analytic Functions
Differential Equations

Hey!!!!

Do you know the binomial theorem? (Pascal's triangle?)

Theorem:

$$(x+y)^n = \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} x^m y^{n-m}$$

A Function of a Complex Parameter

Due to the triangle inequality $|z + w| \leq |z| + |w|$, power series make good sense when plug in complex numbers. In fact, functions that have power series **behave much better** when we use complex numbers.

For Example: Tell me the real reason why the radius of convergence of the power series expression for $\frac{1}{1+x^2}$ centered at $a = 1$ is $\sqrt{2}$.

A Function of a complex Parameter

A complex valued function f is *analytic* with domain an open set U if f has a power series expansion in some disk about each point of U.

Example 1: Let $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Verify $e^{z+w} = e^z e^w$ and that $e^{x+iy} = e^x(\cos(y) + i\sin(y))$.

Example 2

Let $\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$. Verify $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$ and $\cos(z) = \cosh(y)\cos(x) - i\sinh(y)\sin(x)$.

By definition: $\cosh(y) = \frac{e^y+e^{-y}}{2}$ and $\sinh(y) = \frac{e^y-e^{-y}}{2}$

A Function of a complex Parameter

Given an analytic function we can differentiate and integrate via

$$\frac{df}{dz} = \sum_{n=0}^{\infty} n c_n (z-a)^{n-1}$$

$$\int f\,dz = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(z-a)^{n+1} + C$$

A Differential Equations

If f satisfies

$$\sum_{i=0}^{N} a_n(z) \frac{d^n f}{dz^n} = 0,$$

then we say that f is a *solution* to this *differential equation*. An equation in the above special form is called a homogeneous ordinary linear differential equation, a *HOLDE*.

When we try to solve a HOLDE, we must specify initial conditions, namely $f(0) = a_0, \ldots, \frac{d^N f}{dz^{N-1}}(0) = a_{N-1}$

Example 3

Solve

$$\frac{df}{dz} - f = 0$$

$$f(0) = a_0 = 1$$

Solve using power series.

Example 4

Solve

$$\frac{d^2 f}{dz^2} - f = 0$$

$$f(0) = a_0 = 1$$

$$\frac{df}{dz}(0) = a_1 = i$$

Example 5 (Bessel's Equation)

Solve

$$z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} + z^2 f = 0$$

$$f(0) = a_0 = 1$$

$$\frac{df}{dz}(0) = a_1 = 0$$

What is the radius of convergence of the power series you find?