Math 15
176 - 183

176 $\frac{d}{dt}(g \circ \vec{f}) = \nabla g(\vec{f}(t)) \cdot \frac{d\vec{f}}{dt}$

$\frac{\partial g}{\partial x} = y \implies \frac{\partial g}{\partial x}(\vec{f}(t)) = \frac{\partial g}{\partial x}(t^2-1, 2t) = 2t$

$\frac{\partial g}{\partial y} = x \implies \frac{\partial g}{\partial y}(\vec{f}(t)) = \frac{\partial g}{\partial y}(t^2-1, 2t) = t^2-1$

$\frac{d\vec{f}}{dt} = (2t, 2) \implies \frac{dx}{dt} = 2t$ & $\frac{dy}{dt} = 2$

$\implies \frac{d}{dt}(g \circ \vec{f}) = (2t)(2t) + (t^2-1)(2) = 6t^2 - 2$

$\frac{d}{ds}(\vec{f} \circ h) = \frac{d\vec{f}}{dt}(h(s)) \frac{dh}{ds}$

$\frac{d\vec{f}}{dt} = (2t, 2). \implies \frac{d\vec{f}}{dt}(h(s)) = (2\ln s, 2)$

$\frac{dh}{ds} = \frac{1}{s}$

$\implies \frac{d}{ds}(\vec{f} \circ h) = (2\ln s, 2)\frac{1}{s} = \left(\frac{2\ln s}{s}, \frac{2}{s}\right)$

$\frac{\partial}{\partial x}(h \circ g) = \frac{dh}{ds}(g(x,y)) \frac{\partial g}{\partial x}$

$\frac{dh}{ds} = \frac{1}{s} \implies \frac{dh}{ds}(xy) = \frac{1}{xy}$

$\frac{\partial g}{\partial x} = y$

$\implies \frac{\partial}{\partial x}(h \circ g) = \left(\frac{1}{xy}\right)(y) = \frac{1}{x}$

Same procedure for $\frac{\partial}{\partial y}(h \circ g)$

177 $\frac{\partial u}{\partial y} = \frac{du}{dw}(w(x,y))\,\frac{\partial w}{\partial y}$

$\frac{du}{dw} = 2w \Rightarrow \frac{du}{dw}(w(x,y)) = 2xe^y$

$\frac{\partial w}{\partial y} = xe^y$

$\Rightarrow \frac{\partial u}{\partial y} = (2xe^y)(xe^y) = 2x^2e^{2y}$

Similarly for other compositions

178 $(g\circ\vec{f})'(t) = \nabla g(\vec{f}(t))\cdot\vec{f}'(t)$

$(g\circ\vec{f})(3) = \nabla g(\vec{f}(3))\cdot\vec{f}'(3) = \nabla g(1,2,1)\cdot(4,2,-1)$

$= (-1,2,2)\cdot(4,2,-1) = -2$

179 $\frac{d(x,y)}{dr} = \left(\frac{dx}{dr}, \frac{dy}{dr}\right) = \left(\frac{dx}{dt}\frac{dt}{dr}, \frac{dy}{dt}\frac{dt}{dr}\right)$

$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = 2\pi$

@ $r=1$, $t=5$ & $\frac{dt}{dr} = 2$

$\frac{d(x,y)}{dr}(1) = ((2\cdot5)(2), (2\pi)(2)) = (20, 4\pi)$

180 Let $\vec{r}(t)$ be the path of the radiation detector where $\vec{r}(t_0)=(1,2,-1)$

$$\Longrightarrow \frac{d}{dt}\left(R\circ\vec{r}\right) = \nabla R(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} = \nabla R(\vec{r}(t)) \cdot \vec{v}$$

$$\nabla R(x,y,z) = \left(-2x\, e^{-(x^2+y^2+z^2)}, -2y\, e^{-(x^2+y^2+z^2)}, -2z\, e^{-(x^2+y^2+z^2)}\right)$$

$$\Longrightarrow \nabla R(1,2,-1) = \left(-2e^{-6}, -4e^{-6}, 2e^{-6}\right)$$

$$\Longrightarrow \frac{d}{dt}\left(R\circ\vec{r}\right)(0) = \left(-2e^{-6}, -4e^{-6}, 2e^{-6}\right)\cdot(3,-2,1) = 4e^{-6}$$

192 1. $V(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$

$$\vec{F}(x,y,z) = -\nabla V = -\left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right)$$

$$\frac{\partial V}{\partial x} = -x\left(x^2+y^2+z^2\right)^{-3/2}, \quad \frac{\partial V}{\partial y} = -y\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}, \quad \frac{\partial V}{\partial z} = -z\left(x^2+y^2+z^2\right)^{-3/2}$$

$$\vec{F}(x,y,z) = \left(\frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}}\right)$$

Similarly for 2-4.

183 1. $\vec{F}(x,y) = (e^x, \ y\sin y) \overset{?}{=} -\nabla V = \left(-\frac{\partial V}{\partial x}, -\frac{\partial V}{\partial y}\right)$

$\Rightarrow -\frac{\partial V}{\partial x} = e^x \quad \Rightarrow -\int \frac{\partial V}{\partial x}\,dx = \int e^x\,dx$

$\Rightarrow V(x,y) = -e^x + f(y)$

$\Rightarrow -\frac{\partial V}{\partial y} = -\frac{df}{dy} = y\sin y$

$\Rightarrow \int \frac{df}{dy}\,dy = f(y) = \int -y\sin y\,dy$

Integrating by parts, $u = y \quad du = dy \quad dv = -\sin y\,dy \quad v = \cos y$

$\Rightarrow f(y) = y\cos y - \int \cos y\,dy = y\cos y - \sin y + c$

$\Rightarrow V(x,y) = e^x + y\cos y - \sin y$ is a potential for $\vec{F}(x,y)$

Similarly for 2 - 5.