LECTURE OUTLINE
Complex Numbers

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Math 15

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Goals

Introduce Complex Numbers

Geometry of Complex Numbers

Polar Form

We can write a complex number z in *cartesian* or *polar* form

$$z = x + yi = r(\cos(\theta) + i\sin(\theta)) \equiv re^{i\theta},$$

and we say $\arg(z) = \theta$ is z's *argument*, $|z| = r$ is z's *norm*, $x = Re(z)$ is z's *real part*, and $y = Im(z)$ is z's *imaginary part*,

Find $z = \frac{5\sqrt{3}}{2} + i\frac{5}{2}$ norm, argument, real part, imaginary part, and express z in polar form.

Addition

Let $z = x_1 + y_1 i$ and $w = x_2 + y_2 i$. Then $z + w = (x_1 + x_2) + (y_1 + y_2)i$, is simply vector addition.

Add $z = \frac{5\sqrt{3}}{2} + \frac{5}{2}i$ to $w = 1 + i$.

Multiplication

Let $z = r_1 e^{i\theta_1}$ and $w = r_2 e^{i\theta_2}$. Then $zw = (r_1 r_2) e^{i(\theta_1 + \theta_2)}$. Check that if we express this in cartesian form that we are simply "FOILing".

Square $z = \frac{5\sqrt{3}}{2} + \frac{5}{2}i$ and describe the norm and argument of the square.

Multiply $z = \frac{5\sqrt{3}}{2} + \frac{5}{2}i$ and $w = 1 + i$.

Powers

Let $z = re^{i\theta}$ and note $z^n = r^n e^{in\theta}$. This tells us how to take roots, namely

$$(z)^{\frac{1}{n}} = r^{\frac{1}{n}} e^{i\left(\frac{\theta + k2\pi}{n}\right)}$$

for any k from 0 to $n - 1$.

Find the square, cube, 4th and 5th roots of 1. Graph them. (These are called the *roots of unity*.)

The Inverse

Let $z = x + yi = re^{i\theta}$ be a non-zero complex number, then z's inverse is

$$\frac{1}{z} = \frac{x - yi}{x^2 + y^2} = \frac{1}{r}e^{-i\theta}$$

and we say $\bar{z} = x - yi$ is z's *complex conjugate*, and note $z^{-1} = \frac{\bar{z}}{|z|^2}$.

Find the inverse of $z = \frac{5\sqrt{3}}{2} + i\frac{5}{2}$ and express it is polar and cartesian form.

Examples

1. Express $1 + \frac{2}{1-i}$ in the form $a + bi$.
2. Let $z = x + yi$ and find the real and imaginary parts of

$$\frac{z+1}{3z-2}$$

3. Simplify $(1-i)^5$.