LECTURE OUTLINE
Potential

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Math 15

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Goals

The Fundamental Theorem of Line Integrals

Potentials

Review

A force is called *conservative* if the work done by $\vec{F}$ as an object traverses a curve γ depends on only on γ's end points.

Those Caveats: : Suppose $\vec{F}$ is continuous in some open set U such that every pair of points in the set can be connected by a piecewise differentiable continuous path contained in the set. Call such a path *reasonable*. $\vec{F}$ is conservative if given any pair reasonable paths $\gamma_1([a_1, b_1])$ and $\gamma_2([a_2, b_2])$ that satisfy $\gamma_1(a_1) = \gamma_2(a_2)$ and $\gamma_1(b_1) = \gamma_2(b_2)$ we have that $W_{\vec{F}}(\gamma_1) = W_{\vec{F}}(\gamma_2)$.

The Fundamental Theorem of Line Integrals

Theorem: $\vec{F}$ is conservative if and only if $\vec{F} = -\nabla\varphi$.

We call $-\varphi$ the force's *potential*.

Here the set where $-\nabla\varphi$ is continuously differentiable is that open set U from the previous caveat.

The Line Integral Theorem's Proof

Suppose $\vec{F} = -\nabla\varphi$ and $\gamma([a,b])$ is a reasonable path with $\gamma(a) = \vec{A}$ and $\gamma(b) = \vec{B}$. We need to show $W_{\vec{F}}(\gamma)$ depends only on $\vec{A}$ and $\vec{B}$. Well...

$$W_{\vec{F}}(\gamma) = \int_\gamma \vec{F}\cdot d\vec{r} = \int_a^b -\nabla\varphi\cdot\frac{d\gamma}{dt}dt$$

$$= -\int_a^b \frac{d\varphi(\gamma(t))}{dt}dt = -(\varphi(\gamma(b)) - \varphi(\gamma(a)) = \varphi(\vec{A}) - \varphi(\vec{B}).$$

Suppose $\vec{F}$ is conservative, and suppose γ is a reasonable path from $\vec{A}$ (a fixed point) to $\vec{r}$. We will check in class that $\varphi(\vec{r}) = -\int_\gamma \vec{F}\cdot dt$ is a function which satisfies $\vec{F} = -\nabla\varphi$.

An Example of a Potential

Find the force associated to the *gravitational potential*

$$\varphi = -\frac{mMG}{r}.$$

Suppose $r = r_e + h$ and $M = m_e$, and m is your mass. Find c_0 and c_1 in $\varphi(h) \approx c_0 + c_1 h$.

When is there a potential?

Is $\vec{F} = y\hat{i} + y\hat{j} + z^2\hat{k}$ conservative? If so, then find its potential.

Is $\vec{F} = 2x\cos(e^y + x^2)\hat{i} + e^y\cos(e^y + x^2)\hat{j} + 5\hat{k}$ conservative? If so, then find its potential.

Conservation of Energy

We know that our total force as we traverse $\gamma(t)$ satisfies $KE + PE(\gamma(t)) = Constant$. (Suppressed in this is the fact that $PE(\gamma(t))$ may depend on γ.) We have just seen that if our total force is conservative, then

$$KE + \varphi(\vec{r}) = Constant,$$

hence this equation has nothing to do with our choice of γ.

Suppose $M = 1000$, $m = 1$, and $G = 1$, and that we are acted on by gravity with an initial speed 10. Suppose we started at $r = 100$ and find our selves at $r = 50$. What is our current speed?