LECTURE OUTLINE
Solving Differential Equations By Factoring

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Math 15

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Goal

Review: Power Series Method

Differential Operators

Factoring Differential Equations

Summary of Last Night

Theorem: Given function $p(z)$ and $q(z)$ that can be described by power sries that converges for $|z - z_0| \leq R$ and the equation

$$\frac{d^2 f}{dz^2} + p(z)\frac{df}{dz} + q(z)f = 0,$$

the *power series method* will always produce a solution to this equation that can also described a as power series that converges for $|z - z_0| \leq R$ subject any choice of *initial conditions* $f(z_0) = A$ and $\frac{df}{dx}(z_0) = B$.

Furthermore any solution to such an equation is uniquely determined by its initial conditions $f(z_0) = A$ and $\frac{df}{dx}(z_0) = B$.

Other Views

Let $D = \frac{d}{dz}$ (the act of taking a derivative). D^n means perform this action n times. Then our previous equation can be written as

$$D^2 f + p(z)Df + q(z)f = 0.$$

As such we are viewing our differential equation as something we do. $\Delta = D^2 f + p(z)Df + q(z)f$ an example of an *linear differential operator*.

Example 4

Using the $\Delta f = D^2 f + \omega^2$, solve the equation

$$\Delta f = 0$$

$$f(0) = A$$

$$Df(0) = B$$

Example 5

Let $\Delta = D^2 f + \omega^2$. Factor and find two solutions to

$$\Delta f = 0.$$

Big Theorem

Theorem: If $\Delta f = D^2 f + p(z)Df + q(z)f$ and c and d are any constants, then $\Delta(cg + dh) = c\Delta g + d\Delta h$.

Corollary: If $\Delta f = D^2 f + p(z)Df + q(z)f$ and g and h solve the equation $\Delta f = 0$, then $cg + dh$ also solves this equation.

Example 6

Let $\Delta = D^2 + \omega^2$. Solve

$$\Delta f = 0$$

$$f(0) = A$$

$$\frac{df}{dz}(0) = B$$

in two ways.

Using this Corollary

Key Fact: To solve the equation $\Delta f = 0$ for any *initial conditions* $f(0) = A$ and $Df(0) = B$ we can simply find 2 solutions $g(z)$ and $h(z)$ such that for some choice of c and d we can solve $cf(0) + dh(0) = A$ and $cDf(0) + dDh(0) = B$.

Example 7: Damped Harmonic Motion

Try to find all solutions to

$$\frac{d^2 f}{dz^2} + \gamma \frac{df}{dz} + cf = 0$$

with $\gamma > 0$ and $c > 0$.

Critically Damped Harmonic Motion

Use the power series method to find a solution to

$$\frac{d^2 f}{dz^2} + 2\omega \frac{df}{dz} + \omega^2 f = 0$$

$$f(0) = 0$$

$$\frac{df}{dz}(0) = 1.$$

Using the key fact, find the solution to this equation.

Example 7 (Bessel's Equation)

Try to solve

$$z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} + z^2 f = 0$$

$$f(0) = A$$

$$\frac{df}{dz}(0) = B$$

What happens?