6. Suppose we have an ideal pendulum of length 1 meter and mass 1. From rest we impart our pendulum with a speed of $\sqrt{g}\frac{m}{sec}$. (Recall two forces act on such an ideal pendulum, a tension in the $-\hat{r}$ direction and the force of gravity acting in the $-\hat{j}$ direction of magnitude $g\frac{m}{sec^2}$. Assume this g and the g in the above $\sqrt{g}$ are the same.)

a. Let θ be the angle our pendulum makes with $-\hat{j}$, and find the potential energy of our pendulum for each θ,

A solution using polar stuff: (We do not need to use any polar stuff here, I just like it!) Notice since we are on the unit circle $\vec{\gamma}(t) = \hat{r}(t - \frac{\pi}{2})$ for t from 0 to θ parameterizes the needed circular arc. We have that

$$PotntialEnergy(\theta) \equiv PE(\theta) = -Work(\theta) = -\int_0^\theta \vec{F} \cdot \frac{d\vec{\gamma}}{dt} dt$$

where the total force is described in the problem as $\vec{F} = (-g\vec{j} - (Ten)\hat{r})$ and $\frac{d\vec{\gamma}}{dt}(t) = \hat{\theta}(t - \frac{\pi}{2})$. So $PE(\theta)$ can be computed as

$$-\int_0^\theta (-g\vec{j} - (Ten)\hat{r}) \cdot \hat{\theta}(t - \frac{\pi}{2}) dt = g\int_0^\theta \cos(t - \frac{\pi}{2}) dt = g\int_0^\theta \sin(t) dt = g(1 - \cos(\theta))$$

Note: One could also have observed that only the gravitation force contributes to the potential energy in this situation, and recalled that the gravitation potential is $mgh = (1)(g)(1 - \cos(\theta))$.

b. Find the pendulum's speed when it is at an angle θ.

Solution: $\vec{F}$ is the total force acting on the system, hence energy is conserved. In other words, since the mass is 1

$$PE(\theta) + \frac{1}{2}|\vec{v}(\theta)|^2 = Constant \equiv C$$

By plugging $t = 0$, we find $C = 0 + \frac{1}{2}|\sqrt{g}|^2 = \frac{g}{2}$. Moving the potential energy term to the right hand side of the equation, we now have

$$\frac{1}{2}|\vec{v}(\theta)|^2 = \frac{g}{2} - PE(\theta) = \frac{g}{2} - g(1 - \cos(t)) = g(-\frac{1}{2} + \cos(\theta)).$$

So our speed when the pendulum is at an angle θ is $\sqrt{g(-1 + 2\cos(\theta))}$.

Note: We need $(-\frac{1}{2} + \cos(\theta)) \geq 0$ for this square root to make sense. Hence, any θ that fails to satisfy this inequality is physically impossible and our pendulum will fail to have enough "umph" to reach such a non-physical angle.

c. Does our pendulum make a complete circle? (Justify your answer **carefully**.)

Solution: If $\frac{1}{2}|\vec{v}(\theta)|^2 = 0$, then we will have stoped. From part (b), this is equivalent to $(-\frac{1}{2} + \cos(\theta)) = 0$ having a solution. Recalling that $\cos(\pm\frac{\pi}{3}) = \frac{1}{2}$, we see that our pendulum stops at when $\theta = \frac{\pi}{3}$ and so our pendulum will **fail** to make a full revolution.

7. Suppose we choose our units so that the force of gravity exerted by a massive object located at the origin of our three-dimensional axes on a small object of mass 1 located a distance r from the origin has magnitude $\frac{2}{9}\frac{1}{r^2}$ and acts directly towards the origin. I **tell** you that this is a conservative force.

(a.1) Parameterize a particle that travels from $(1,0,0)$ to $(1/2,0,0)$ in a straight line and at a constant speed.

Solution: Well $\vec{\gamma}(t) = (1-\frac{t}{2},0,0)$ goes from $(1,0,0)$ to $(1/2,0,0)$ as t ranges from 0 to 1.

(a.2) Use your parameterization to compute the work done by gravity in going from $(1,0,0)$ to $(1/2,0,0)$.

Solution: $r(t) = \sqrt{(1-\frac{t}{2})^2+0^2+0^2} = 1-\frac{t}{2}$ for t in $[0,1]$, so the force due to gravity is

$$\vec{F} = -\frac{2}{9}\frac{1}{r^2}\hat{\imath} = -\frac{2}{9}\left(1-\frac{t}{2}\right)^{-2}\hat{\imath}.$$

Now the work is $\int_0^1 \vec{F}\cdot\frac{d\vec{\gamma}}{dt}dt$ with $\frac{d\vec{\gamma}}{dt} = -\frac{1}{2}\hat{\imath}$, hence the work done by $\vec{F}$ equals

$$\int_0^1 \frac{1}{9}(1-\frac{t}{2})^{-2}dt = \frac{4}{9}\int_0^1 (2-t)^{-2}dt = \frac{4}{9}(2-t)^{-1}\Big|_0^1 = \frac{4}{9}-\frac{2}{9} = \frac{2}{9}.$$

(b) What is the work done in going from $(1,0,0)$ to $(1/2,0,0)$ along $\vec{r}(t) = ((1-t)^{2/3},0,0)$ for $0 \le t \le 1-\frac{\sqrt{2}}{4}$?

Solution: $\vec{r}(t)$ is another parameterization of the line from $(1,0,0)$ to $(1/2,0,0)$, and the work is parameterization independent, so the work equals $\frac{2}{9}$.

Note: **Since** I am told that the force is conservative, I know that the work is in fact entirely independent of the path.

(c) Is it possible that the force of gravity is the only force acting on your parameterized curve from part (a) or the curve $\vec{r}(t)$ from part (b)? Justify your answers **carefully**.

Solution: We must check whether either path is consistent with our equation of motion $\vec{F} = m\vec{a} = \vec{a}$ (since our mass is assumed 1). Well in part $(a.2)$ we found $\vec{F} = -\frac{2}{9}(1-\frac{t}{2})^{-2}\hat{\imath}$ as we move along our line from $(a.1)$. But the acceleration

$$\frac{d^2\vec{\gamma}}{dt^2} = 0 \neq -\frac{2}{9}(1-\frac{t}{2})^{-2}\hat{\imath},$$

so $\vec{\gamma}(t)$ must be acted on by **forces other than gravity** to experience this motion. Using the path $\vec{r}(t) = ((1-t)^{2/3},0,0)$, we see that $\vec{F} = -\frac{2}{9}\frac{1}{r^2}\hat{\imath} = -\frac{2}{9}(1-t)^{-4/3}\hat{\imath}$ and that our acceleration equals

$$\frac{d^2\vec{r}}{dt^2} = (\frac{d^2}{dt^2}(1-t)^{2/3})\hat{\imath} = (\frac{d}{dt}(-2/3)(1-t)^{-1/3})\hat{\imath} = -\frac{2}{9}(1-t)^{-4/3}\hat{\imath}.$$

So this path could indeed be describing the trajectory of particle under the influence of gravity alone.