CLASS 24, 5/20/2011, FOR MATH 13, SPRING 2011

1. More examples of the Divergence Theorem in use

Example. Let $\mathbf{F} = \langle 2x + x^2, -2xy, z+2\rangle$, and let S be the boundary of the cone $0 \le z \le 2 - \sqrt{x^2+y^2}$. Calculate the flux of $\mathbf{F}$ through S.

A direct calculation would probably be pretty messy (there would be two separate integrals, one for the base of the cone, and the other for the curved part, both of which would reduce to double integrals which would require polar coordinates), so let's instead try the Divergence Theorem.

Since $\nabla \cdot \mathbf{F} = 2 + 2x - 2x + 1 = 3$, the Divergence Theorem says

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iiint_E \nabla\cdot\mathbf{F}\, dV = \iiint_E 3\, dV = 3\iiint_E dV = 3V(E).$$

Since E is a solid cone with base 2 and height 2, its volume is $8\pi/3$, so the surface integral we wanted to calculate is equal to 8π.

Example. The idea behind the previous example is that we can replace the calculation of a surface integral across a non-closed surface with the calculation of another surface integral over a different non-closed surface, which together with the original surface form a closed surface which encases a solid E, and then also calculate the integral of $\nabla\cdot\mathbf{F}$ over E. While this seems like we're replacing one problem with another which involves more work (after all, we need to calculate an additional triple integral), this sometimes turns out to involve less calculation if the original surface is very complicated, but the secondary surface and $\nabla\cdot\mathbf{F}$ turn out to be simple.

For example, let S be the surface given by the the part of the sphere $x^2+y^2+z^2 = 4$ with $z \le 1$, with outward pointing orientation, and let $\mathbf{F} = \langle e^z, xz, x^2+y^2\rangle$. If we wanted to directly calculate the surface integral of $\mathbf{F}$ across S, we would need to parameterize S using spherical coordinates (not pleasant), and the expression for $\mathbf{F}\cdot\mathbf{n}$ would be very complicated.

Instead, we will carry out the technique described above. Let's complete S to a closed surface by adding a cap S' which consists of points $x^2+y^2 \le 3, z = 1$. (You have flexibility in choosing how you want to complete S, but you want to make the choice which will result in the simplest possible calculations. Determining the proper choice is something you gain with experience.) To ensure that S and S' have compatible orientations, we need S' to have orientation pointing upwards.

Furthermore, we quickly see that $\nabla\cdot\mathbf{F} = 0$. If we let E be the solid that S, S' enclose, we have

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS + \iint_{S'} \mathbf{F}\cdot\mathbf{n}\, dS = \iiint_E \nabla\cdot\mathbf{F}\, dV = \iiint_E 0\, dV = 0.$$

Therefore,

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_{S'} -\mathbf{F}\cdot\mathbf{n}\, dS.$$

Although the left hand side is difficult to directly calculate, the right hand side is easy to calculate. Since $\mathbf{n} = \langle 0, 0, 1\rangle$ on S' (since S' is parallel to the xy plane, and has upwards orientation), we have $\mathbf{F} \cdot \mathbf{n}\, dS = x^2 + y^2$ on S'. Furthermore, S' is easily parameterized by $\mathbf{r}(u, v) = \langle u, v, 1\rangle, u^2 + v^2 \leq 3$. Therefore,

$$\iint_{S'} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{D} u^2 + v^2\, dA$$

where D is the domain $u^2 + v^2 \leq 3$. This looks like an integral we should use polar coordinates to solve; letting $u = r\cos\theta, v = r\sin\theta$, we have

$$\iint_{D} u^2 + v^2\, dA = \int_0^{2\pi} \int_0^{\sqrt{3}} r^2 r\, dr\, d\theta = \int_0^{2\pi} \frac{r^4}{4}\Big|_{r=0}^{r=\sqrt{3}} d\theta = 2\pi \frac{9}{4} = \frac{9\pi}{2}.$$

Therefore, the original integral we wanted to calculate (which was over S instead of S') is equal to $-9\pi/2$.

2. The Divergence Theorem and physics

The Divergence Theorem is also a very useful mathematical tool in electromagnetism. Recall that if we place a particle of charge Q at the origin of $\mathbb{R}^3$, the electric field it generates can be described by the equation

$$\mathbf{E} = \frac{Q}{d^3}\mathbf{d},$$

where $\mathbf{d}$ is the vector $\langle x, y, z\rangle$. One can directly check that $\nabla \cdot \mathbf{E} = 0$ at every point except the origin, where $\nabla \cdot \mathbf{E}$ is not defined, since $\mathbf{E}$ is not defined there.

Suppose we want to calculate the flux of the electric field generated by such a charge across a closed surface S whose interior E contains the origin. We cannot apply the divergence theorem directly to E, because $\mathbf{E}$ is not defined at every point of E (recall that we need $\mathbf{E}$ to be C^1 at every point of E, which it cannot be if $\mathbf{E}$ is not even defined at every point of E). However, we can apply the divergence theorem to the solid E' we obtain by cutting out a little sphere centered at the origin. Strictly speaking, using the divergence theorem as we stated it requires that the boundary of E' consist of one piece, but we can circumvent this by cutting E' up into several pieces and applying the divergence theorem to each. (The same trick works with Greens' Theorem.) In any case, the boundary of E' consists of S (with outward orientation) together with the small sphere, which we'll call S'. Notice that S' has orientation pointing towards the origin, because the boundary of E should have orientation pointing out from E.

It's probably not possible to evaluate the flux of $\mathbf{E}$ across the surface S, which could have an irregular shape, but it is easy to calculate the flux of $\mathbf{E}$ across S'. Since the boundary of E' consists of S and S', and $\nabla \cdot \mathbf{E} = 0$ on E', the divergence theorem tells us

$$\iint_{S} \mathbf{E} \cdot \mathbf{n}\, dS + \iint_{S'} \mathbf{E} \cdot \mathbf{n}\, dS = \iiint_{E'} \nabla \cdot \mathbf{E}\, dV = \iiint_{E'} 0\, dV = 0.$$

Therefore, if we can calculate the flux of $\mathbf{E}$ across S', we will know what the flux of $\mathbf{E}$ across S is.

Since S' is a sphere, the unit normals for S' have a simple form. Suppose S' has radius r. Recall that if $\mathbf{d} = \langle x, y, z\rangle$, then the unit normal to S' at $\mathbf{d}$ is equal to $-\mathbf{d}/|\mathbf{d}|$. (The negative sign appears since S' has orientation pointing inwards, and $\mathbf{d}$ is evidently orthogonal to S' at the point $\mathbf{d}$.) Therefore,

$$\mathbf{E}\cdot\mathbf{n} = \frac{Q}{d^3}\mathbf{d}\cdot -\frac{\mathbf{d}}{|\mathbf{d}|} = \frac{-Q}{d^2} = \frac{-Q}{r^2}.$$

On the other hand, S' is a sphere of radius r, so we have

$$\iint\limits_{S'} \mathbf{E}\cdot\mathbf{n}\,dS = \iint\limits_{S'} \frac{-Q}{r^2}\,dS = \frac{-Q}{r^2}\iint\limits_{S'} dS = \frac{-Q}{r^2}4\pi r^2 = -4\pi Q.$$

This tells us that

$$\iint\limits_{S} \mathbf{E}\cdot\mathbf{n}\,dS = -\iint\limits_{S'} \mathbf{E}\cdot\mathbf{n}\,dS = 4\pi Q.$$

In other words, the electric flux through a closed surface S does not depend on the shape of S, but only on whether it contains the origin or not.

On the other hand, if S does not contain the origin, then we can directly apply the divergence theorem to the divergence-free vector field $\mathbf{E}$ on the interior of S, and we see that

$$\iint\limits_{S} \mathbf{E}\cdot\mathbf{n}\,dS = 0.$$

Because the electric field of several charges is additive, we have shown that the flux of an electric field $\mathbf{E}$ generated by some charge distribution across a surface S is given by the formula

$$\iint\limits_{S} \mathbf{E}\cdot\mathbf{n} = 4\pi Q_{\text{in}},$$

where Q_{in} is the total amount of charge enclosed by S. This is sometimes known as *Gauss' Law.* Remember that this formula only holds for the electric field generated by some stationary charge distribution, so it will not apply in most situations in this class, but when studying electromagnetism, many of the questions deal precisely with the electric field generated by some charge distribution.

Example. As an application of Gauss' Law, consider the following question. Suppose we have an infinite sheet of charge, with uniform density 1, on the xy plane. What is the strength of the electric field at an arbitrary point (x, y, z)?

We will use Gauss' Law, in combination with simple geometric symmetry arguments, to determine the electric field that this distribution of charge generates. (The direct way to calculate an electric field generated by some charge distribution is to sum the contribution of each charge, which in this case would involve calculating an integral since we have a continuous as opposed to discrete charge distribution.) First, notice that $\mathbf{E}$ must be of the form $\langle 0, 0, f(z)\rangle$ for some function f which only depends on z; the reason this is true is because of symmetry. There should be no x or y component because any contribution to the x component and y component of the electric field from a point $(x, y, 0)$ will be cancelled out by the contribution from $(-x, -y, 0)$. Furthermore, every point with the same z coordinate will have the same electric field because the plate of charge is infinite and symmetric in the x, y directions.

With this in mind, let's examine a box whose vertices are $(\pm a, \pm b, \pm c)$. On the one hand, this box encloses $(2a)(2b) = 4ab$ units of charge. On the other hand, the electric flux through this box is given by the sum of the electric fluxes through the top and bottom (the faces with $z = \pm c$), because the flux through the other four sides equals 0. The flux through the top is given by $f(c)ab$, so altogether the flux through this box is given by $2f(c)ab$. Therefore, Gauss' Law tells us

$$2f(c)ab = 16\pi ab \Rightarrow f(c) = \frac{1}{8\pi}.$$

Interesting enough, the strength of the electric field does not depend on the distance from the plate at all!

Another application of the Divergence Theorem to physics is that it relates the integral form of Gauss' Law to the differential form of Gauss' Law. Gauss' Law as we've described it is in the integral form

$$\iint_S \mathbf{E} \cdot \mathbf{n}\, dS = 4\pi Q_{\text{in}}.$$

The Divergence Theorem tells us that the left hand side is also equal to

$$\iiint_E \nabla \cdot \mathbf{E}\, dV = 4\pi Q_{\text{in}}.$$

On the other hand, we can write Q_{in} as the integral of the charge density function over E. Therefore,

$$\iiint_E \nabla \cdot \mathbf{E}\, dV = 4\pi \iiint_E \rho\, dV,$$

where ρ is the charge density function which describes a distribution of charge. The only way this formula can be true for every solid E is if $\nabla \cdot \mathbf{E} = 4\pi\rho$, which is one of Maxwell's Equations. Hopefully these examples suggest why the Divergence Theorem is a tool of fundamental mathematical importance in classical electromagnetism.