CLASS 11, 4/20/2011, FOR MATH 13, SPRING 2011

1. Spherical coordinates

We now briefly examine another coordinate system which is sometimes convenient when computing triple integrals. *Spherical coordinates* are defined in the following way: a point (x, y, z) has spherical coordinates (ρ, θ, ϕ) if

$$x = \rho \sin\phi \cos\theta, y = \rho \sin\phi \sin\theta, z = \rho \cos\phi.$$

One can check, using the identity $\sin^2\theta + \cos^2\theta = 1$, that $x^2 + y^2 + z^2 = \rho^2$; therefore, $|\rho|$ is just the distance of a point from the origin. The meaning of the other two angles is not as obvious, but do have a natural geometric interpretation.

The angle ϕ, sometimes called the *inclination angle* or *polar angle*, measures the angle the line segment from the origin to the point in question makes with the positive z-axis. Therefore, if $\phi = 0$, the corresponding point lies somewhere on the positive z-axis; if $\phi = \pi/2 = 90°$, then the point lies somewhere on the xy plane (since this plane is orthogonal to the z-axis), and if $\phi = \pi = 180°$, then the point lies somewhere on the negative z-axis. With this interpretation of ϕ, it is evident that we want to restrict ϕ to lie between 0 and π.

Once we have determined ρ and ϕ, we have also determined z, but x, y are still undetermined. If you project (x, y, z) onto the xy plane, you get the point $(x, y, 0)$. Define $r = \rho \sin\phi$; then r is the distance of $(x, y, 0)$ from the origin. The angle θ, sometimes called the *azimuthal angle*, is the angle which makes $x = r\cos\theta, y = r\sin\theta$; evidently, this is the angle the line segment connecting $(x, y, 0)$ to the origin makes with the positive x-axis, just like when we considered polar or cylindrical coordinates. We want to restrict θ to lie in between 0 and 2π.

Examples.

- The graphs of equations $\rho = C, C$ some positive constant, are spheres, with center at the origin and radius ρ. This explains why these coordinates are called cylindrical coordinates.
- The graph of an equation $\phi = C$, $0 \le C \le \pi$, will in general be a cone of varying width, centered around the z-axis. If C is very small, then the cone is very thin, while if $C = \pi/2$, we actually get the xy plane instead of a cone.
- The graph of an equation $\theta = C, 0 \le \theta \le 2\pi$, is a plane which contains the z-axis, and is orthogonal to the xy plane. This is very closely related to the fact that the graphs of $\theta = C$ in polar coordinates are lines passing through the origin.
- Spherical coordinates are probably already familiar to you. The system of latitude and longitude used to identify points on the surface of the Earth are very closely related to ϕ and θ. For example, pretend the Earth is a perfect sphere and has $\rho = 1$. Then the set of points with ϕ constant form a line of latitude, since we will be looking at all points on the surface of the Earth with $z = \phi$ constant. There is a slight difference in the actual numbers used for latitude as opposed to ϕ, but this is merely cosmetic. For example, the North Pole corresponds to $\phi = 0$, which is at $90°N$ latitude. The equator is $0°$, which corresponds to $\phi = \pi/2$. Similarly, lines of longitude are given by θ constant. This picture of spherical coordinates as essentially being the system of latitude and longitude also allow us to see what a 'spherical

rectangle' looks like. The rectangular prism $R = [x_1, x_2] \times [y_1, y_2] \times [z_1, z_2]$ is really just a rectangular prism in rectangular coordinates. In cylindrical coordinates, a 'cylindrical rectangle' takes the shape of an angular piece of the solid between two concentric cylinders. In spherical coordinates, a 'spherical rectangle' takes the shape of a solid between two concentric spheres, demarcated by some line of latitude and some line of longitude.

- What is the shape of the solid given by $0 \le \rho \le 2, 0 \le \phi \le \pi, 0 \le \theta \le 3\pi/2$? Evidently we obtain three fourths of a sphere of radius 2; the part lying over the fourth quadrant of the xy plane is cut out.
- We remark that there is no uniform notation for spherical coordinates, even to this modern day. The terminology we have chosen, with ϕ for the polar angle and θ for the azimuth, is common amongst mathematicians and is used by the textbook. However, in other fields (physics, engineering, etc.), it is possible that the roles of ϕ and θ are interchanged. When consulting other sources for formulas related to spherical coordinates, be sure to check which notation is being used!

What is the integration formula connecting spherical coordinates to an iterated integral? If a region E is given by spherical inequalities $0 \le a \le \rho \le b, \alpha \le \theta \le \beta, \gamma \le \phi \le \delta$, then

$$\iiint_E f(x, y, z)\, dV = \int_\gamma^\delta \int_\alpha^\beta \int_a^b f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi.$$

In other words, we need to multiply the integrand by $\rho^2 \sin\phi$ when converting to spherical coordinates. There is a similar expression for regions E which are defined by more general spherical inequalities, where some of the bounds of integrations may be functions in the variables not yet integrated, instead of constants.

Example. Use spherical coordinates to show that the volume of a sphere with radius R is equal to $4\pi R^3/3$.

A sphere of radius R, say E, is described using spherical coordinates by $0 \le \rho \le R, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi$. Therefore, the volume of the sphere is given by the triple integral

$$\iiint_E dV = \int_0^\pi \int_0^{2\pi} \int_0^R \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi = \left(\int_0^\pi \sin\phi \, d\phi\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^R \rho^2 \, d\rho\right) = 2\pi(2)\frac{R^3}{3} = \frac{4\pi R^3}{3}.$$

Example. Consider a sphere. What proportion of the sphere satisfies $0 \le \phi \le \phi_0$?

Example. We can just assume the radius of our sphere is 1. If we let E be the region defined by $0 \le \rho \le 1, 0 \le \phi \le \phi_0, 0 \le \theta \le 2\pi$, then we are asking for the ratio of the volume of E to the volume of a sphere with radius 1. The latter is $4\pi/3$. For the former, we want to evaluate the integral

$$\iiint_E 1 \, dV = \int_0^{2\pi} \int_0^{\phi_0} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi.$$

We evaluate this in the usual way to get

$$2\pi \cdot \frac{1}{3} \cdot (1 - \cos\phi_0),$$

so dividing this by $4\pi/3$ gives a proportion $\dfrac{1 - \cos\phi_0}{2}$. One interpretation of this calculation is that the above fraction gives the proportion of the Earth's mass which has latitude above a certain fixed latitude, assuming the density of the Earth is constant. (This is not true, but it is almost true, and density is probably not constant as a function of radius due to the different components of the Earth's interior, but because of the spherical symmetry of the problem this won't matter.)

Example. Consider a solid, of uniform density, which fills the space E between $x^2 + y^2 + z^2 \leq 4, x^2 + y^2 + z^2 \leq 9$, and also has $z \geq 0$. What is the center of mass of this solid?

Symmetry immediately indicates that $\overline{x} = \overline{y} = 0$. To calculate $\overline{z}$, we first should calculate the mass of this solid. Let us assume that $\rho(x, y, z) = 1$; then the mass is the volume of the solid, which is equal to

$$\frac{4}{3}\pi(3^3 - 2^3)\frac{1}{2} = \frac{38\pi}{3}.$$

The moment about the xy plane is equal to

$$\iiint_E z\, dV.$$

The region E is described by spherical coordinates $2 \leq \rho \leq 3, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi/2$. Therefore, this integral is equal to

$$\int_0^{\pi/2} \int_0^{2\pi} \int_2 3(\rho \cos\phi)(\rho^2 \sin\phi)\, d\rho\, d\theta\, d\phi.$$

We use the trick which tells us that this integral is equal to the product of three single integrals, since the integrand is a product of r^3 with $\cos\phi \sin\phi$ and 1:

$$\left(\int_0^{\pi/2} \cos\phi \sin\phi\, d\phi\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_2^3 \rho^3\, d\rho\right) = \left(\frac{\sin^2\phi}{2}\Big|_0^{\pi/2}\right) 2\pi \left(\frac{\rho^4}{4}\Big|_2^3\right) = \frac{65\pi}{4}.$$

Therefore, $\overline{z}$ is given by

$$\frac{65\pi}{4} \cdot \frac{3}{38\pi} = \frac{195}{152}.$$