CLASS 10, 4/18/2011, FOR MATH 13, SPRING 2011

1. Interchanging order of integration: triple integrals

Recall that if a two-dimensional region D was both type I and type II, we could interchange the order of integration, but in so doing had to rewrite the inequalities which defined D appropriately. In a similar way, we can occasionally change the order of integration over three-dimensional regions E, but because we are working with three-dimensional regions which are harder to visualize, it is often much more difficult to determine the correct bounds of integration. We will consider the region in the last example of the previous class, which is relatively easy to work with, to see how this might work in a simple situation.

Example. (Skip this example in class) In the previous example, we considered the iterated integral

$$\int_0^2 \int_0^{1-x/2} \int_0^{3-3x/2-3y} dz\, dy\, dx,$$

which represented the volume of a certain tetrahedron. Write down the iterated integrals obtained by changing the order of integration to $dy\, dx\, dz$, and $dx\, dz\, dy$.

As in the two-dimensional case, we begin by drawing a sketch of the region E which we are integrating over. In this problem, one description of it is as the set of points satisfying the inequalities

$$0 \leq x \leq 2, 0 \leq y \leq 1 - \frac{x}{2}, 0 \leq z \leq 3 - \frac{3x}{2} - 3y.$$

Another description which is less dependent on the ordering of the variables is given by the part of the first octant which is bounded by the plane $z = 3 - 3x/2 - 3y$, or equivalently the plane $3x + 6y + 2z = 6$.

If we want to change the order of integration to $dy\, dx\, dz$, we will integrate with respect to y first. Therefore, we need to determine inequalities which y satisfies, as a function of x and z. A lower bound for y is clearly given by $y \geq 0$, and an upper bound is given by $y \leq 1 - x/2 - z/3$, because possible values for y are bounded from above by the y-coordinates of points on the plane given by $3x + 6y + 2z = 6$.

Now we need to determine the bounds on x, which should be in terms of z. To do so, we project the region E onto the xz plane. In this example, it is fairly easy to see that the projection is the triangle in the xz plane whose vertices have (x, z) coordinates $(0, 0), (0, 3), (2, 0)$. Therefore, this triangle is bounded by the lines $x = 0, z = 0, z = 3 - 3x/2$. Therefore, x satisfies the inequalities $0 \leq x \leq 2 - 2z/3$. Finally, z is bounded by $0 \leq z \leq 3$. Therefore, if we change the order of integration we obtain the integral

$$\int_0^3 \int_0^{2-2z/3} \int_0^{1-x/2-z/3} dy\, dx\, dz.$$

One can check that this integral is equal to the original integral in question.

If we want to change the order of integration to $dx\, dz\, dy$, we begin by finding bounds on x in terms of y, z. Clearly $x \geq 0$. An upper bound for x is given by the plane $3x + 6y + 2z = 6$, or $x = 2 - 2y - 2z/3$. To determine bounds for z in terms of y, we project E onto the yz plane. We see that this is the triangle with vertices whose (y, z) coordinates are given by

$(0,0),(1,0),(0,3)$. Therefore, $0 \leq y$, and $y \leq 1 - z/3$. Finally, bounds for y are given by $0 \leq y \leq 1$. Then the integral we obtain is

$$\int_0^1 \int_0^{1-z/3} \int_0^{2-2y-2z/3} dx\, dz\, dy.$$

In this example we are in the very fortunate situation where the projection of E onto any of the three coordinate planes (xy, xz, yz) is equal to the intersection of E with those coordinate planes. However, this need not be the case in general. The next example gives a simple and natural situation where this is not so, and also provides a nice transition into our next topic.

Example. Consider the region E satisfying the inequalities $\sqrt{x^2+y^2} \leq z \leq 1$. Write down iterated integrals which equal the triple integral of a function $f(x,y,z)$ over the region E with order $dz\,dy\,dx$ and $dx\,dy\,dz$.

We begin by drawing a sketch of E. The surface $z = \sqrt{x^2+y^2}$ is a cone with vertex at the origin, which gets wider as z increases. Therefore, the region given by $\sqrt{x^2+y^2} \leq z \leq 1$ is an inverted cone whose vertex lies at the origin, and whose upper boundary is given by a disc $x^2+y^2 \leq 1, z = 1$.

If we want to write an iterated integral with order of integration $dz\,dy\,dx$, we start by determining bounds on z in terms of x, y. These are basically given to us in this problem, as $\sqrt{x^2+y^2} \leq z \leq 1$. To determine the bounds on y, x, we examine the projection of E onto the xy plane. In this situation, we see that the projection of E is the disc $x^2+y^2 \leq 1$. Therefore, y satisfies the inequalities $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$, while x satisfies the inequalities $-1 \leq x \leq 1$. Of course, the intersection of E with the xy plane is just the origin, so in this example the projection of E onto the xy plane is something larger than the intersection of E with the xy plane. In any case, the iterated integral in question is

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^1 f(x,y,z) dz\, dy\, dx.$$

If we want an iterated integral with order of integration $dx\,dy\,dz$, we first start by determining what inequality x must satisfy as a function of y, z. The x coordinates of points in E are evidently bounded by the boundary of the cone, so will be bounded by the functions of x we obtain when we solve for x in $x^2+y^2 = z^2$. This yields $x = \pm\sqrt{z^2-y^2}$, so $-\sqrt{z^2-y^2} \leq x \leq \sqrt{z^2-y^2}$. To determine bounds for y, z, we begin by projecting E onto the yz plane. In this case, one can see that the projection will be a triangle whose vertices have coordinates $(0,0), (-1,1), (1,1)$. The inequalities which y satisfies are thus given by $-z \leq y \leq z$. Finally, $0 \leq z \leq 1$, so the iterated integral in question is

$$\int_0^1 \int_{-z}^{z} \int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} f(x,y,z)\, dx\, dy\, dz.$$

Therefore, we see that writing down an iterated integral equal to a triple integral can be a fairly lengthy and difficult process (and we haven't even done any integrating yet!). We first need to determine the inequalities that the first variable we will integrate satisfy, in terms of the other two remaining variables. This often involves identifying the surfaces which bound the region E, and then solving for the variable in question in terms of the other two variables.

We then need to determine the projection of the region E onto the plane spanned by the remaining two variables. In general this is a very difficult question, and is only solvable in very special situations. You certainly should not expect to be able to solve this question for 'arbitrary' regions E, so if you are asked to write down an iterated integral of a region E then there must be a way to somewhat easily determine the projection of E onto one of the three coordinate planes. Once this projection is determined, writing down the bounds for the remaining two variables reduces to a two-dimensional question which we have encountered many times when evaluating double integrals.

2. Cylindrical coordinates

We will now take a look at evaluating triple integrals using coordinate systems different from rectangular coordinates. This is exactly the analogue of when we studied polar coordinates with double integrals. We begin with *cylindrical coordinates*, which are very closely related to polar coordinates.

A point (x, y, z), in rectangular coordinates, has cylindrical coordinates (r, θ, z), if $x = r\cos\theta, y = r\sin\theta$. That is, we extend polar coordinates to three dimensions by simply tacking on an additional z coordinate. The reason why these coordinates are called cylindrical coordinates is clear if we look at the surfaces determined by equations of the form $r = C$, for various constants C. These are cylinders which are centered around the z-axis.

We want a formula which relates an integral over rectangular coordinates to a corresponding integral over cylindrical coordinates. If a region E in (x, y, z) space is given by cylindrical inequalities $0 \le a \le r \le b, \alpha \le \theta \le \beta (0 \le \beta - \alpha \le 2\pi), z_1 \le z \le z_2$, then we have an inequality of integrals

$$\iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_a^b \int_{z_1}^{z_2} f(r\cos\theta, r\sin\theta, z) r\, dz\, dr\, d\theta.$$

Notice the similarity of this formula to the formula relating integration over polar coordinates to rectangular coordinates in two dimensions. There is the same factor of r appearing in the integrand, and we replace each of x, y, z in the integrand with $r\cos\theta, r\sin\theta, z$, respectively.

Of course, a similar formula holds true in the more general case where z might be a function of r, θ, and r a function of θ.

Cylindrical coordinates are really useful in problems involving not only cylinders, but any situations where expressions of the form $x^2 + y^2$ appear.

Example. Consider the cone described in the last problem – the set of points described by the inequalities $\sqrt{x^2 + y^2} \le z \le 1$. Write down a triple integral in cylindrical coordinates which equals the volume of this cone.

Notice that the inequalities on r, z are given by $0 \le r \le z \le 1$, since $x^2 + y^2 = r^2$. In particular, the inequalities for z are $r \le z \le 1$. The inequalities on θ are given by $0 \le \theta \le 2\pi$, since any cross-section of the cone by a plane $z = C$ is a disc. From the first set of inequalities we see that $0 \le r \le 1$.

$$\iiint_E dV = \int_0^{2\pi} \int_0^1 \int_1^r r\, dz\, dr\, d\theta.$$

Notice that this integral is much easier to calculate than the various iterated integrals we found in the previous example:

$$\int_0^{2\pi}\int_0^1\int_r^1 r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r(1-r)\,dr\,d\theta = \int_0^{2\pi}\left(\frac{r^2}{2}-\frac{r^3}{3}\Big|_{r=0}^{r=1}\right)d\theta = \int_0^{2\pi}\frac{1}{6} = \frac{\pi}{3}.$$

Suppose that this cone is a solid with density given by $\rho(x,y,z) = z$, and we want to calculate the mass of the solid. Then we want to evaluate

$$\iiint_E z\,dV = \int_0^{2\pi}\int_0^1\int_r^1 rz\,dz\,dr\,d\theta.$$

Again, this integral is easy to evaluate (contrast it to the corresponding integral you would need to evaluate in rectangular coordinates):

$$\int_0^{2\pi}\int_0^1\int_r^1 rz\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r\frac{z^2}{2}\Big|_{z=r}^{z=1}dr\,d\theta = \int_0^{2\pi}\int_0^1 r(\frac{1}{2}-\frac{r^2}{2})\,dr\,d\theta = \int_0^{2\pi}\frac{r^2}{4}-\frac{r^4}{8}\Big|_{r=0}^{r=1}d\theta = \frac{\pi}{4}.$$

Example. Consider the solid E bounded by the surfaces $z = \sqrt{x^2+y^2}, z = 2 - x^2 - y^2$. Write down a triple integral equal to the volume of this solid, and evaluate it.

First, we begin by sketching these two surfaces. The first surface is a cone (as we have seen), while the second surface is a elliptic (even circular, in this case) paraboloid which is up-side down. Furthermore, the cone forms the bottom boundary of the surface while the paraboloid forms the top. Evidently, this solid looks somewhat like an ice-cream cone.

We want to determine the projection of E onto the xy plane. To do this, we should perhaps start by finding the intersection of the two surfaces. We set both functions of x, y equal to each other and obtain $\sqrt{x^2+y^2} = 2 - x^2 - y^2$. If we let $r = \sqrt{x^2+y^2}$ (which is natural as we are going to eventually end up using cylindrical coordinates anyway), this equation becomes $r = 2 - r^2$. This is a quadratic for r, and the solutions to this equation are $r = 1, -2$. We discard the $r = -2$ solution since $r \geq 0$, and so the two surfaces intersect when $r = 1$. Furthermore, when $r = 1, z = r = 2 - r^2 = 1$, so the intersection of these two surfaces is a circle given by the equations $x^2 + y^2 = 1, z = 1$.

Amongst all points in E, r is maximal at the boundary of this circle, so the projection of E onto the xy plane is the disc $x^2 + y^2 \leq 1$. In cylindrical coordinates, this corresponds to inequalities $0 \leq \theta \leq 2\pi, 0 \leq r \leq 1$. Therefore, the triple integral which equals the volume of this solid is

$$\iiint_E dV = \int_0^{2\pi}\int_0^1\int_r^{2-r^2} r\,dz\,dr\,d\theta.$$

We evaluate this integral:

$$\int_0^{2\pi}\int_0^1\int_r^{2-r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r(2-r^2-r)\,dr\,d\theta = 2\pi\left(r^2-\frac{r^3}{3}-\frac{r^4}{4}\Big|_{r=0}^{r=1}\right) = 2\pi\left(1-\frac{1}{3}-\frac{1}{4}\right) = \frac{5\pi}{6}.$$

Of course, we could have instead just calculated a double integral over the disc $x^2+y^2 \leq 1$, using polar coordinates, to find this volume. However, setting up a triple integral has the advantage of letting us also calculate quantities like mass, moments, moment of inertia, etc., of a solid in the shape of E – we would not need to adjust the bounds of integration, only the integrand.