# Conservative Vector Fields and Path Independence

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## Definitions

- A path $C$ is **simple** if it doesnt cross itself.

- A region $D$ is **open** if it doesnt contain any of its boundary points.

- A region $D$ is **connected** if we can connect any two points in the region with a path that lies completely in $D$.

## Simply-Connected

A region $R$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ is **simply-connected** if it consists of a single connected piece and if every simple closed curve $C$ in $R$ can be continuously shrunk to a point while remaining in $R$ throughout the deformation.

# Path-Independent Line Integrals

**Definition:** A continuous vector field $\mathbf{F}$ has **path-independent line integrals** if

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s}$$

for any two simple, piecewise $C^1$, oriented curves in the domain of $\mathbf{F}$ with the same endpoints.

## Path-Independent Property

**Theorem:** Let $\mathbf{F}$ be continuous vector field. Then $\mathbf{F}$ has a path-independent line integrals if and only if

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = 0$$

for every piecewise $C^1$, simple, closed curve $C$ in the domain of $\mathbf{F}$.

# Conservative Vector Fields

$\mathbf{F}$ is a **conservative vector field** if there is a scalar function $f$ such that

$$\mathbf{F} = \nabla f$$

The function $f$ is called a **potential function** for the vector field.

## Test for a vector field to be conservative

Let $\mathbf{F}$ be a vector field of class $C^1$ whose domain is simply-connected region $R$ in either $\mathbb{R}^2$ or $\mathbb{R}^3$. Then $\mathbf{F} = \nabla f$ for some scalar-valued function $f$ of class $C^2$ on $R$ if and only if

$$\nabla \times \mathbf{F} = \mathbf{0}$$

for all points of $R$.

## Path-Independence and Conservative fields

If $\mathbf{F}$ is a continuous vector field on an open connected region $D$ and if $\int_C \mathbf{F} \cdot d\mathbf{s}$ is independent of path (for any path in $D$) then $\mathbf{F}$ is a conservative vector field on $D$.

## Fundamental Theorem of Line Integrals

Suppose that $C$ be a $C^1$ oriented path given by $\mathbf{c}(t)$, $a \leq t \leq b$. And suppose that $f$ is a function whose gradient vector, $\nabla f$, is continuous on $C$. Then,

$$\boxed{\int_{\mathbf{c}} \nabla f \cdot d\mathbf{s} = f(\mathbf{c}(b)) - f(\mathbf{c}(a))}$$

Note that $\mathbf{c}(a)$ represents the initial point on $C$ while $\mathbf{c}(b)$ represents the final point on $C$.