Change of Variables for triple integrals

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Coordinate Transformations in dimension 3

A C^1 function $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ that transforms the uvw-space to the xyz-space.

Linear Transformations map in 3 dimensions parallelepipeds to parallelepipeds

In a similar way we can define for every 3×3 matrix A with nonzero determinant. A linear transformation.

The tranformation $T(u, v, w) = A \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ maps parallelepipeds to parallelepipeds.

If $T(D^*) = D$ then
$Volume(D) = |det(A)| \cdot Volume(D^*)$

Important examples of a nonlinear transformation

Cylindrical Coordinates:

$$(x, y, z) = T(r, \theta, z) = (r \cos\theta, r \sin\theta, z)$$

Spherical Coordinates:

$$\begin{aligned}(x, y, z) &= T(\rho, \phi, \theta) \\ &= (\rho \sin(\phi) \cos(\theta), \rho \sin(\phi) \sin(\theta), \rho \cos(\phi))\end{aligned}$$

Jacobian in 3D

Coordinate Transformation:

$T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}.$$

Jacobian for cylindrical and spherical coordinates

Cylindrical:

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = r$$

Spherical:

$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \rho^2 \sin(\phi)$$

Change of Variables in Triple Integrals

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$,
$T(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w))$ be a coordinate transformation from uvw-space to xyz-space that maps W^* to W. Then

$$\iiint_W f(x, y)\, dxdydz$$

$$= \iiint_{W^*} f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| dudvdw$$

Triple Integrals in Cylindrical Coordinates

$$\iiint_W f(x, y, z)\, dxdydz$$

$$= \iiint_{W^*} f(r\cos\theta, r\sin\theta, z)\mathbf{r}\, drd\theta dz$$

$dV = dxdydz$ in Cartesian coordinates
$dV = \mathbf{r} drd\theta dz$ in cylindrical coordinates.

Triple Integrals in Spherical Coordinates

$$\iiint_W f(x,y,z)\,dxdydz$$

$$= \iiint_{W^*} f(x(\rho,\phi,\theta), y(\rho,\phi,\theta), z(\rho,\phi,\theta))\rho^{\mathbf{2}} \sin\phi \, d\rho d\phi d\theta$$

$dV = dxdydz$ in Cartesian coordinates
$dV = \rho^{\mathbf{2}} \sin\phi \, d\rho d\phi d\theta$ in spherical coordinates.