# Change of Variables for Double Integrals

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## Coordinate Transformations in dimension 2

A $C^1$ function $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that transforms the $uv$-plane to the $xy$-plane.

## Linear Transformation

A **Linear Transformation** $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is defined by

$$\begin{aligned} T(u, v) &= (au + bv, cu + dv) \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \end{aligned}$$

Here $a, b, c$, and $d$ are scalar constants.

## Linear Transformations map in 2 dimensions parallelograms to parallelograms

If $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation and $det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \neq 0$ ($T$ is invertible), then

(1) $T$ takes parallelograms to parallelograms and the vertices of a parallelogram map to vertices.

(2) If $T(D^*) = D$, then

Area$(D) = |det(A)| \cdot$ (Area$(D^*)$.

## Important examples of a nonlinear transformation

### Polar Coordinates:

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$T(r, \theta) = (r \cos\theta, r \sin\theta)$$

## The Jacobian of a Transformation in 2D

The **Jacobian** of the tranformation $T$ is the determinant of the derivative matrix $DT(u,v)$.

$$\frac{\partial(x,y)}{\partial(u,v)} = det(DT(u,v)) = det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

## Change of Variables in Double Integrals

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$,

$$T(u,v) = (x(u,v), y(u,v))$$

be a coordinate transformation from $uv$-plane to $xy$-plane that maps $D^*$ to $D$. Then

$$\iint_D f(x,y)dxdy = \iint_{D^*} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv$$

# Double Integrals in Polar Coordinates

$$\iint_D f(x,y)\,dxdy = \iint_{D^*} f(r\cos\theta, r\sin\theta)\mathbf{r}\,drd\theta$$

Note: Jacobian of $T(r,\theta) = (r\cos\theta, r\sin\theta)$ is just $r$.

$dA = dxdy$ in Cartesian coordinates
$dA = \mathbf{r}drd\theta$ in polar coordinates.