Exercises: Double and Triple Integrals
Solutions
Math 13, Spring 2010

1. Consider the iterated integral

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x,y,z)dy\,dz\,dx.$$

   (a) Rewrite this integral as an equivalent iterated integral in the order $dy\,dx\,dz$.

   (b) Rewrite the integral in the order $dz\,dx\,dy$.

Solution. The triple integral is an integral over the solid W in $\mathbb{R}^3$ that consists of all points (x,y,z) with

$$\begin{aligned} 0 &\le x \le 1 \\ 0 &\le z \le 1-x^2 \\ 0 &\le y \le 1-x. \end{aligned}$$

To obtain the $dy\,dx\,dz$ integral we only need to change the order of the two *outer* integrals, which involves studying the "shadow" of W as a region D in the xz-plane. This region D is consists of points (x,z) with

$$\begin{aligned} 0 &\le x \le 1 \\ 0 &\le z \le 1-x^2. \end{aligned}$$

If you draw a diagram of this region, you see that it can also be described as

$$\begin{aligned} 0 &\le z \le 1 \\ 0 &\le x \le \sqrt{1-z}. \end{aligned}$$

This is sufficient to re-write the triple integral as

$$\int_0^1 \int_0^{\sqrt{1-z}} \int_0^{1-x} f(x,y,z)dy\,dx\,dz.$$

In order to get the $dz\,dx\,dy$ integral we need to analyze the three dimensional solid W, by sketching a diagram. Because the two outer integrals are $dx\,dy$, we then need to project W to find its shadow onto the xy-plane. It turns out that this shadow in the xy plane is the region bounded by the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$.

But for the present example there happens to be a way to see this without drawing the 3D diagram. Let's concentrate on the two *inner* integrals in the original triple integral,

$$\int_0^{1-x^2} \int_0^{1-x} f(x, y, z) dy\, dz.$$

We see that the bounds for these two integrals are *independent of* y *and* z. This means that, as far as the $dy\, dz$ integral is concerned, all the bounds are *constants* (because x is a constant here), and we are therefore integrating over a *rectangle* in the yz plane. It is therefore allowed to simply reverse these two integrals to get

$$\int_0^{1-x} \int_0^{1-x^2} f(x, y, z) dz\, dy.$$

If we do this the triple integral becomes

$$\int_0^1 \int_0^{1-x} \int_0^{1-x^2} f(x, y, z) dz\, dy\, dx.$$

We now have dz on the inside, which is what we need. Now all we need to do is reverse the order of the two *outer* integrals, namely the $dy\, dx$ integrals. We need to sketch the diagram of the shadow of W on the xy-plane, but we don't need the full 3D picture. We can see what this shadow is by looking at the outer bounds of the $dz\, dy\, dx$ integral. The shadow of W on the xy plane consists of points with

$$0 \le x \le 1$$
$$0 \le y \le 1 - x.$$

This is the region bounded by the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$. If we slice this triangle up differently we see that it can also be described by

$$0 \le y \le 1$$
$$0 \le x \le 1 - y.$$

So with these new bounds for the two outer integrals we get the $dz\, dx\, dy$ integral as

$$\int_0^1 \int_0^{1-y} \int_0^{1-x^2} f(x, y, z) dz\, dx\, dy.$$

2. The solid W in $\mathbb{R}^3$ is bounded by the sphere $x^2 + y^2 + z^2 = 1$ and the xy-plane. The mass density of the solid W is given by the function $f(x, y, z) = 1 + x^2 + y^2 + z^2$ (in units of mass per units of volume). Calculate the total mass of W.

Solution. The problem is best handled by using spherical coordinates. The solid W is then described as the collection of points (ρ, ϕ, θ) such that

$$\begin{aligned}0 &\leq \rho \leq 1,\\ 0 &\leq \phi \leq \pi/2,\\ 0 &\leq \theta \leq 2\pi.\end{aligned}$$

The mass density function is expressed in the (ρ, ϕ, θ) variables as

$$f(x, y, z) = 1 + x^2 + y^2 + z^2 = 1 + \rho^2.$$

Finally, the volume element in spherical coordinates is given by the standard formula

$$dV = dx\, dy\, dz = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

Putting it all together we obtain the iterated integral

$$\int_0^1 \int_0^{\pi/2} \int_0^{2\pi} (1 + \rho^2)\rho^2 \sin\phi\, d\theta\, d\phi\, d\rho.$$

This integral can be evaluated easily because (1) no functions of ρ, ϕ or θ appear as lower or upper bounds in any of the three integral signs and (2) the integrand $(1+\rho^2)\rho^2 \sin\phi$ separates as a *product* of $(1+\rho^2)\rho^2$ and $\sin\phi$, each of which is a one-variable functions for ρ and ϕ respectively. In such a case the triple integral is equal to a product of three ordinary (single) integrals,

$$\int_0^1 (1 + \rho^2)\rho^2 d\rho \int_0^{\pi/2} \sin\phi\, d\phi \int_0^{2\pi} 1\, d\theta.$$

The calculation of these integrals is as follows.

$$\begin{aligned}\int_0^1 (1 + \rho^2)\rho^2 d\rho &= \int_0^1 \rho^2 + \rho^4 d\rho = \left[\frac{1}{3}\rho^3 + \frac{1}{5}\rho^5\right]_0^1 = \frac{8}{15},\\ \int_0^{\pi/2} \sin\phi\, d\phi &= [-\cos\phi]_0^{\pi/2} = 1,\\ \int_0^{2\pi} 1\, d\theta &= 2\pi.\end{aligned}$$

The final result is

$$\iiint_W f\, dV = \frac{8}{15} \cdot 1 \cdot 2\pi = \frac{16}{15}\pi.$$

3. Evaluate the following integral by changing to polar coordinates,

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{1}{\sqrt{x^2+y^2}}\, dy\, dx.$$

Solution. First we analyze the region D in the xy plane over which we integrate. It is given as

$$\begin{aligned} 0 &\le x \le 2 \\ 0 &\le y \le \sqrt{2x-x^2}. \end{aligned}$$

This is the region bounded below by the x axis ($y = 0$) and above by the graph of $y = \sqrt{2x-x^2}$, or $y^2 = 2x - x^2$. We need to convert these two curves to polar coordinates. The positive x axis corresponds to $\theta = 0$. The equation $y^2 = 2x - x^2$ is equivalent to $x^2 + y^2 = 2x$. Using $x^2 + y^2 = r^2$ and $x = r\cos\theta$ this turns into $r^2 = 2r\cos\theta$ or

$$r = 2\cos\theta.$$

By carefully studying the diagram of the region D, we see that θ ranges over the values $0 \le \theta \le \pi/2$ (because the entire region is contained in the first quadrant), while for a fixed value of θ the radius r ranges over the interval $0 \le r \le 2\cos\theta$. We therefore have the following bounds for the integral

$$\int_0^{\pi/2} \int_0^{2\cos\theta} \cdots\, dr\, d\theta.$$

Secondly, we need to convert the function (the integrand) to polar coordinates. Using $x^2 + y^2 = r^2$ we find

$$f(x, y) = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}.$$

Thirdly, we have to choose the correct Jacobian for the area form $dA = dy\, dx$. For polar coordinates there is the standard formula

$$dA = dx\, dy = r\, dr\, d\theta.$$

Assembling all the pieces we find

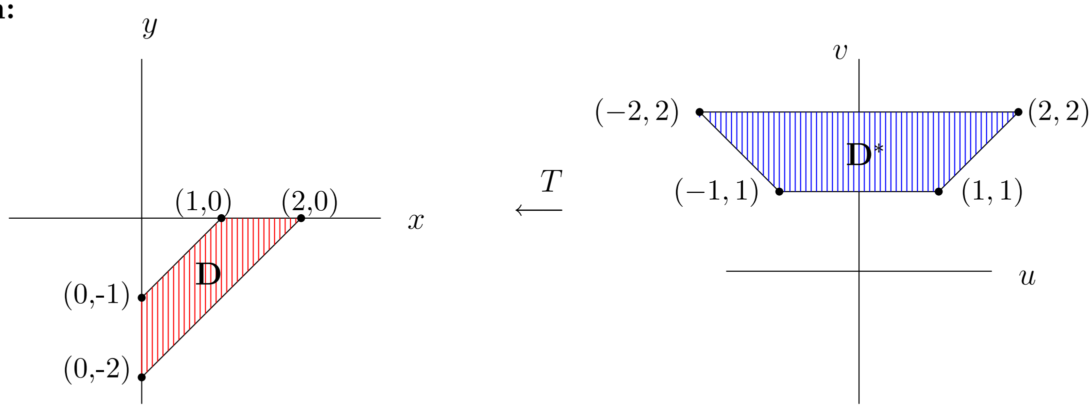
$$\begin{aligned} \int_0^{\pi/2} \int_0^{2\cos\theta} \frac{1}{r} r\, dr\, d\theta &= \int_0^{\pi/2} \int_0^{2\cos\theta} 1\, dr\, d\theta \\ &= \int_0^{\pi/2} 2\cos\theta\, d\theta \\ &= [2\sin\theta]_0^{\pi/2} \\ &= 2. \end{aligned}$$

4. Evaluate the integral

$$\iint_D e^{\frac{x+y}{x-y}}\,dA$$

Where D is the region bounded by the trapezoid with vertices $(1,0)$, $(2,0)$, $(0,-2)$ and $(0,-1)$.

Solution:



In this problem we choose the substitution

$$\begin{aligned} u &= x+y \\ v &= x-y \end{aligned}$$

This is a linear transformation, which means that it maps parallelograms to parallelograms and vertices of parallelograms to vertices of parallelograms. If we solve for x and y. So the vertices map to vertices. This means that we have the following correspondence:

(x,y)	(u,v)
$(1,0)$	$(1,1)$
$(2,0)$	$(2,2)$
$(0,-1)$	$(-1,1)$
$(0,-2)$	$(-2,2)$

The inverse transformation is obtained by solving for x and y and we will use this to find the Jacobian

$$\begin{aligned} x &= \frac{u+v}{2} \\ y &= \frac{u-v}{2} \end{aligned}$$

Using this we compute the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{pmatrix} = -\frac{1}{2}$$

Now the bounds for D^* can be computed from the four vertices in the uv-plane (see the trapezoid in the right-handside):

$$\begin{aligned} -v &\le u \le v \\ 1 &\le v \le 2 \end{aligned}$$

$$\begin{aligned} \iint_D e^{\frac{x+y}{x-y}}\, dA &= \int_1^2 \int_{-v}^{v} e^{\frac{u}{v}} \left|-\frac{1}{2}\right| dudv \\ &= \int_1^2 \frac{v}{2} e^{\frac{u}{v}} |_{u=-v}^{u=v}\, dv \\ &= \int_1^2 \frac{v}{2}(e - e^{-1}) dv \\ &= (e - e^{-1}) \frac{v^2}{2} |_{v=1}^{v=2} \\ &= \frac{3}{4}(e - e^{-1}) \end{aligned}$$

5. Evaluate the integral

$$\iiint_W (x^2 + y^2 + 2z)\, dV$$

where W is the region that lies below the paraboloid $z = 25 - x^2 - y^2$, inside the cylinder $x^2 + y^2 = 4$ and above the xy-plane.

Solution:

The best way to do this problem is to switch to cylindrical coordinates.

The bounds of integration are:

$$\begin{aligned} 0 &\le z \le 25 - r^2 \\ 0 &\le r \le 2 \\ 0 &\le \theta \le 2\pi \end{aligned}$$

Hence,

$$\begin{aligned}
\iiint_W (x^2+y^2+2z)\,dV &= \int_0^{2\pi}\int_0^2\int_0^{25-r^2} (r^2+2z)r\,dzdrd\theta \\
&= \int_0^{2\pi}\int_0^2\int_0^{25-r^2} (r^3+2rz)\,dzdrd\theta \\
&= \int_0^{2\pi}\int_0^2 (r^3z+z^2r)|_{z=0}^{z=25-r^2}\,drd\theta \\
&= \int_0^{2\pi}\int_0^2 (r^3(25-r^2)+(25-r^2)^2r)\,drd\theta \\
&= \int_0^{2\pi}\int_0^2 r(25-r^2)(r^2+25-r^2)\,drd\theta \\
&= \int_0^{2\pi}\int_0^2 625r-25r^3\,drd\theta \\
&= \int_0^{2\pi} \frac{625}{2}r^2-\frac{25}{4}r^4|_{r=0}^{r=2}d\theta \\
&= 2\pi(1150) \\
&= 2300\pi
\end{aligned}$$

6. Evaluate the integral

$$\iint_D 30xy\,dA$$

where D is the region bounded by $y=x$ and $y=x^2$.

Solution: This is a type 3 region and we have a choice on the order of integration. You should do both ways to check that indeed we do get the same answer. In this solution I will choose the bounds

$$\begin{aligned}
x^2 &\le y \le x \\
0 &\le x \le 1
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\int_0^1 \int_{x^2}^{x} 30xy\, dy dx &= \int_0^1 15xy^2|_{y=x^2}^{y=x}\, dx \\
&= \int_0^1 15x^3 - 15x^5\, dx \\
&= \frac{15}{4}x^4 - \frac{15}{6}x^6|_{x=0}^{x=1} \\
&= \frac{15}{4} - \frac{5}{2} \\
&= \frac{5}{4}
\end{aligned}$$