Math 12 Fall 2007 Final Exam

Instructor (circle one): Chernov, Pauls

Friday December 7, 2007

11:30-2:30 PM Kemeny 008

PRINT NAME: ______________________________

Instructions: This is a closed book, closed notes exam. **Use of calculators is not permitted. You must justify all of your answers to receive credit.**
You have **three hours**. Do any 10 out of 11 problems. If you wish you can solve all 11 problems, then we will disregard the problem with the lowest score from this list.
Please do all your work on the paper provided.

The Honor Principle requires that you neither give nor receive any aid on this exam.

Grader's use only:

1. ________ /12

2. ________ /12

3. ________ /12

4. ________ /12

5. ________ /12

6. ________ /12

7. ________ /12

8. ________ /12

9. ________ /12

10. ________ /12

11. ________ /12

Total: ________ /120

1. Let $\mathbf{F}(x, y, z) = -\frac{1}{3}y^3\mathbf{i} + \frac{1}{3}x^3\mathbf{j} + e^{z^3}\mathbf{k}$ be a vector field. Let C be the oriented curve parametrized as $\mathbf{r}(t) = \langle \cos t, \sin t, \cos^2 t + 2\sin^2 t \rangle$, where $t \in [0, 2\pi]$. Compute $\int_C \mathbf{F} \cdot d\mathbf{S}$. Hint: note that the curve lies on the paraboloid surface $z = x^2 + 2y^2$.

2. Let $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ be a vector field on $\mathbb{R}^3$. Assume that all the partial derivatives of all orders of the component functions P, Q, R are continuous. Prove that $\text{div}(\text{curl}\, F)$ is equal to zero at all points.

3. Let C be a smooth curve given by a smooth vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, with $\mathbf{r} : [a, b] \to \mathbb{R}^3$. Let $f(x, y, z)$ be a differentiable function of three variables defined on the whole $\mathbb{R}^3$ whose gradient vector field ∇f is continuous everywhere. Prove that $\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$.

4. Let $f(x,y) = (x-1)^2 + y^2$ be a function and let $D = \{(x,y) | x^2 + y^2 \leq 1\}$ be a domain. Find the absolute maximum and the absolute minimum of the function f in the domain D.

5. Let

$$\mathbf{F}(x,y,z) = P_1(x,y,z)\mathbf{i} + Q_1(x,y,z)\mathbf{j} + R_1(x,y,z)\mathbf{k}$$

and

$$\mathbf{G}(x,y,z) = P_2(x,y,z)\mathbf{i} + Q_2(x,y,z)\mathbf{j} + R_2(x,y,z)\mathbf{k}$$

be vector fields on $\mathbb{R}^3$ such that all the partial derivatives of all orders of the component functions $P_1, P_2, Q_1, Q_2, R_1, R_2$ are continuous. Prove that $\operatorname{curl}(\mathbf{F}+\mathbf{G}) = \operatorname{curl}\mathbf{F} + \operatorname{curl}\mathbf{G}$.

6. Find the value of

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

in the following cases:

(a)

$$\mathbf{F} = \left\langle \frac{1}{2}xy^2 + x^2y + \sin(x^2), \frac{1}{2}x^2y + \frac{1}{3}x^3 + e^y \right\rangle$$

and C is the unit square in the (x,y)-plane oriented **clockwise.**

(b)

$$\mathbf{F} = \left\langle \frac{1}{2}xy^2 + x^2y, \frac{1}{2}x^2y + \frac{1}{3}x^3 \right\rangle$$

and C is any curve connecting $(0,0)$ to $(1,1)$.

7. Find

$$\iint_{\Sigma} \mathbf{F} \cdot d\mathbf{S}$$

where $\mathbf{F} = y^2\ \mathbf{i} + x\ \mathbf{j} + z\ \mathbf{k}$ and Σ is the closed, positively oriented surface which is the union of $z = 4 - x^2 - y^2$ for $0 \leq x^2 + y^2 \leq 4$ and the disk of radius 2 in the (x,y)-plane.

8. Let

$$\mathbf{G} = \langle x + 2y + z^2, x - y + z, x^2 + y^2 \rangle$$

(a) Find $\mathbf{F} = \text{curl } \mathbf{G}$.

(b) Suppose Σ_1 and Σ_2 are two surfaces with common boundary C (as shown in the figure). Equip Σ_1, Σ_2 and C with orientations so that C is positively oriented with respect to both Σ_1 and Σ_2. Place the orientation vectors for the surfaces Σ_1 and Σ_2 at the marked points p and q and draw the orientation on the curve C.

Prove that

$$\iint_{\Sigma_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{\Sigma_2} \mathbf{F} \cdot d\mathbf{S}$$

9. Let f be a continuously differentiable function defined on the entire (x, y)-plane and let $\mathbf{u} = \langle a, b \rangle$ be a vector in the plane. Prove that

$$D_{\mathbf{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u}$$

10. Let

$$f(x,y) = \begin{cases} \dfrac{x^4 + 3y^4}{x^3 + y^3} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) \neq (0,0) \end{cases}$$

Using the limit definition of the partial derivatives, find $f_x(0,0)$ and $f_y(0,0)$.

11. Let S be the sphere of radius a centered at the origin. Find the vector equation for the tangent plane to the sphere at a point (x_0, y_0, z_0).