Math 11
Fall 2007
Practice Problem Solutions

Here are some problems on the material we covered since the second midterm. This collection of problems is not intended to mimic the final in length, content, or difficulty.

The final exam will concentrate on material covered since the second midterm, but there will also be problems on earlier material.

1. True or False:

   (a) The function

   $$\vec{r}(t) = \vec{a} + t(\vec{b} - \vec{a}) \qquad 0 \leq t \leq 1$$

   parametrizes the straight line segment from $\vec{a}$ to $\vec{b}$.

   TRUE. This is a standard way to parametrize a line segment.

   (b) If the coordinate functions of $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ have continuous second partial derivatives, then $curl(div(\vec{F}))$ equals zero.

   FALSE. The divergence of $\vec{F}$ is a scalar function, so its curl is not even defined.

   (c) Putting together the two different vector forms of Green's Theorem, we can see that if D is a region satisfying the hypotheses of the theorem, and P and Q are functions satisfying the hypotheses of the theorem, we must have

   $$\int_{\partial D} (P, Q) \cdot \hat{T}\, ds = \int_{\partial D} (P, Q) \cdot \hat{N}\, ds$$

   Here $\hat{T}$ denotes the unit tangent vector to a curve, and $\hat{N}$ the unit normal vector, so the integral on the left is the usual line integral of $\vec{F} = (P, Q)$ along ∂D, and the integral on the right is the integral representing the flux of $\vec{F} = (P, Q)$ across ∂D.

   FALSE: The two different vector forms of Green's Theorem do deal with these two line integrals, but they are equal to different double integrals over D.

(d) For any vector field $\vec{F}$ in $\mathbb{R}^3$ all of whose coordinate functions have continuous first and second partial derivatives, we have that $div(curl(\vec{F})) = 0$.

TRUE: This is one of the things Clairaut's Theorem tells us; another is that, for a scalar function f with continuous first and second partial derivatives, $curl(grad(f)) = 0$.

(e) If the vector field $\vec{F}$ is conservative on the open region D then line integrals of $\vec{F}$ are path-independent on D, regardless of the shape of D.

TRUE: The Fundamental Theorem of Line Integrals tells us this. It is necessary for D to have no holes if we want to use the fact that $curl(\vec{F}) = \vec{0}$ on D to tell us that $\vec{F}$ is conservative on D.

(f) If $\vec{F}$ is any vector field, then $curl(\vec{F})$ is a conservative vector field.

FALSE: We know the curl of a conservative vector field must be $\vec{0}$, so if this were true then we would always have $curl(curl(\vec{F})) = \vec{0}$ for any vector field $\vec{F}$. But this is not true. (Try computing $curl(curl(xz,\, yz,\, 0))$.)

2. (a) Find a potential function f for the vector field

$$\vec{F}(x, y) = (2x + 2y,\, 2x + 2y).$$

A potential function is just a function f such that $\vec{F} = \nabla f$.

SOLUTION: There are several ways to go about finding a potential function. The organized antidifferentiation method is as follows:
If $\nabla f = \vec{F}$ we must have

$$f_x = 2x + 2y \tag{1}$$

$$f_y = 2x + 2y \tag{2}$$

Starting with Equation 1 and integrating with respect to x (treating y as a constant) we get

$$f_x = 2x + 2y$$

$$f = x^2 + 2xy + C(y) \tag{3}$$

Note that the "constant" of integration C can actually be a function of y, as we are treating y as a constant. Now we differentiate this equation with respect to y to get

$$f_y = 2x + C'(y)$$

Comparing this to Equation 2 we see that we must have

$$f_y = 2x + 2y$$

$$f_y = 2x + C'(y)$$
$$C'(y) = 2y$$

which we can achieve by letting

$$C(y) = y^2.$$

(If we were trying to satisfy an initial condition, say $f(1, 1) = 0$, we would say $C(y) = y^2 + B$, where B is a constant, and later use the initial condition to solve for B. But we aren't, so we just go ahead.) Plugging this back into Equation 3 we see we can set

$$\boxed{f(x, y) = x^2 + 2xy + y^2.}$$

The less organized antidifferentiation method begins with the same two equations

$$f_x = 2x + 2y$$
$$f_y = 2x + 2y$$

and then integrates the first with respect to x and the second with respect to y to get

$$f = x^2 + 2xy + C(y)$$

$$f = 2xy + y^2 + A(x)$$

Examining these two expressions for terms in common, we guess that

$$f = x^2 + 2xy + y^2$$

will work. Since there is an element of inspiration in this last step, it is *important* to check that this is the correct answer by computing ∇f to make sure that in fact $\nabla f = \vec{F}$.

A third way to compute f is to use the Fundamental Theorem of Line Integrals. We know that if γ is any path from $(0,0)$ to (a,b), and if $f(0,0) = 0$ then

$$\int_\gamma \nabla f(x,y) \cdot d\vec{r} = f(a,b) = f(0,0) = f(a,b).$$

We can let γ consist of the straight line from $(0,0)$ $(a,0)$ followed by the straight line from $(a,0)$ to (a,b). Then we have

$$f(a,b) = \int_\gamma \nabla f(x,y) \cdot dr = \int_\gamma \langle 2x+2y,\, 2x+2y\rangle \cdot dr =$$

$$\int_\gamma (2x+2y)\,dx + (2x+2y)\,dy = \int_0^a (2x+0)\,dx + \int_0^b (2a+2y)\,dy =$$

$$a^2 + 2ab + b^2.$$

(b) Verify the Fundamental Theorem of Line Integrals for $\int_C \vec{F} \cdot d\vec{r}$ in the case

$$\vec{F}(x,y) = (2x+2y,\, 2x+2y)$$

and C is the portion of the positively oriented circle $x^2 + y^2 = 25$ from $(5,0)$ to $(3,4)$.

SOLUTION: To verify the Fundamental Theorem of Line Integrals is to check that it is true in this case. The Fundamental Theorem of Line Integrals tells us that, in this case, if $\vec{F} = \nabla F$, then

$$\int_C \vec{F} \cdot d\vec{r} = f(3,4) - f(5,0).$$

We have already found f such that $\vec{F} = \nabla f$, namely

$$f(x,y) = x^2 + 2xy + y^2 = (x+y)^2$$

so that

$$f(3,4) - f(5,0) = (3+4)^2 = (5+0)^2 = 24,$$

so to verify the Fundamental Theorem of Line Integrals in this case, we must compute

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \langle 2x + 2y,\, 2x + 2y \rangle \cdot d\vec{r}$$

and see that the answer is also 24.
To compute the line integral we parametrize C by

$$\vec{r}(t) = \langle 5\cos t,\, 5\sin t \rangle$$

for $0 \leq t \leq \theta$, where θ is the angle whose cosine is $\frac{3}{5}$ and whose sine is $\frac{4}{5}$. Then we have

$$d\vec{r} = \langle -5\sin t,\, 5\cos t \rangle \; dt$$

$$\int_C \vec{F} \cdot d\vec{r} =$$

$$\int_0^\theta \langle 10\cos t + 10\sin t,\, 10\cos t + 10\sin t \rangle \cdot \langle -5\sin t,\, 5\cos t \rangle \; dt =$$

$$\int_0^\theta 50(\cos^2 t - \sin^2 t)\, dt = \int_0^\theta 50\cos(2t)\, dt =$$

$$50\frac{\sin(2t)}{2}\Big|_0^\theta = 50\frac{\sin(2\theta)}{2} = 50\cos\theta\sin\theta = 50\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) = 24.$$

3. Find $\int_C \vec{F}(x,y)\, d\vec{r}$ where

$$\vec{F}(x,y) = \left(ye^{xy} + \cos x,\, xe^{xy} + \frac{1}{y^2+1} \right)$$

and C is the portion of the curve $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{2}$.

SOLUTION: We can check that $\vec{F}$ is conservative, by checking that

$$\frac{\partial}{\partial x}\left(xe^{xy} + \frac{1}{y^2+1} \right) = \frac{\partial}{\partial y}\left(ye^{xy} + \cos x \right) = xye^{xy}$$

Therefore this problem calls for the Fundamental Theorem of Line Integrals. Using the same method as in the last problem we see that

$$\vec{F}(x,y) = \nabla f(x,y)$$

where

$$f(x,y) = e^{xy} + \sin x + \tan^{-1} y.$$

Therefore

$$\int_C \vec{F}(x,y)\, d\vec{r} = f\left(\frac{\pi}{2}, 1\right) - f(0,0) =$$

$$\left(e^{\frac{\pi}{2}} + \sin\frac{\pi}{2} + \tan^{-1} 1\right) - \left(e^0 + \sin 0 + \tan^{-1} 0\right) =$$

$$e^{\frac{\pi}{2}} + 1 + \frac{\pi}{4} - 1 = \boxed{e^{\frac{\pi}{2}} + \frac{\pi}{4}}$$

An alternative method of using the fact that $\vec{F}$ is conservative, rather than finding f and applying the Fundamental Theorem of Line Integrals, is to use path independence. That is, instead of computing the line integral of $\vec{F}$ along C, we can compute the line integral of $\vec{F}$ along any other curve with the same endpoints, say the line segment from $(0,0)$ to $\left(\frac{\pi}{2}, 0\right)$ followed by the line segment from $\left(\frac{\pi}{2}, 0\right)$ to $\left(\frac{\pi}{2}, 1\right)$.

4. The temperature at a point in space is given by the function

$$T(x,y,z) = z^2 - xy.$$

Heat flows from regions of high temperature to regions of low temperature, and the rate of heat flow is proportional to the rate at which temperature changes. That is, heat flow (in appropriate units) is given by

$$\vec{F}(x,y,z) = -\nabla T(x,y,z).$$

The rate at which heat flows across a surface S is given by the flux of the heat flow $\vec{F}$ across S,

$$\iint_S \vec{F} \cdot d\vec{S}.$$

If S is the surface given in cylindrical coordinates by

$$z = \theta \qquad r \leq 1 \qquad 0 \leq \theta \leq 2\pi$$

oriented so the unit normal vector slants upwards, find the rate at which heat flows across S.

Don't try anything fancy here. Just parametrize the surface and compute the flux.

SOLUTION: We can use cylindrical coordinates to parametrize the surface, letting u be r and v be θ. Then our parametrization is

$$\langle x, y, z\rangle = \vec{r}(u,v) = \langle u\cos v,\ u\sin v,\ v\rangle \qquad 0 \le u \le 1 \qquad 0 \le v \le 2\pi$$

and our surface integral giving the flux is

$$\iint_S \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^1 \vec{F}(u\cos v,\ u\sin v,\ v)\cdot(\vec{r}_u \times \vec{r}_v)\,du\,dv$$

Now we have

$$\vec{F}(x,y,z) = -\nabla T(x,y,z) = -\langle -y,\ -x,\ 2z\rangle$$

$$\vec{F}(u\cos v,\ u\sin v,\ v) = \langle u\sin v,\ u\cos v,\ -2v\rangle$$

$$\vec{r}_u = \langle \cos v,\ \sin v,\ 0\rangle$$

$$\vec{r}_v = \langle -u\sin v,\ u\cos v,\ 1\rangle$$

$$\vec{r}_u \times r_v = \langle \sin v,\ -\cos v,\ u\rangle$$

$$\vec{F}(u\cos v,\ u\sin v,\ v)\cdot(\vec{r}_u \times \vec{r}_v) = u(\sin^2 v - \cos^2 v - 2v)$$

$$\iint_S \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^1 \vec{F}(u\cos v,\ u\sin v,\ v)\cdot(\vec{r}_u \times \vec{r}_v)\,du\,dv =$$

$$\int_0^{2\pi}\int_0^1 u(\sin^2 v - \cos^2 v - 2v)\,du\,dv = \boxed{-2\pi^2}$$

5. Find

$$\iint_S \vec{F}\cdot d\vec{S}$$

where S is the conical surface

$$z^2 = x^2 + y^2 \qquad 0 \le z \le 1$$

oriented so the unit normal vector slants downwards, and $\vec{F}$ is the function

$$\vec{F}(x, y, z) = (x + \tan^{-1}(y^2),\ -y + \sec(x + z),\ z^2).$$

Note that S is not a closed surface. Nevertheless, there is a better way to do the problem than brute force.

SOLUTION: Sometimes we can apply Gauss's Theorem (the Divergence Theorem) to compute the surface integral over a surface that is not closed, by viewing that surface as *part* of the boundary of a solid region.

In this case, let E be the region above the surface S and below the plane $z = 1$. That is, E is the region described in cylindrical coordinates by $r \leq z \leq 1$ (a solid cone). The positively oriented boundary of E consists of two pieces, the surface S with the given orientation, and the disc S' described by $x^2 + y^2 \leq 1$, $z = 1$, oriented so its unit normal vector points upwards, $\hat{n} = \hat{k} = \langle 0, 0, 1\rangle$. Now Gauss's Theorem tells us

$$\int\int\int_E div(\vec{F})\, dV = \int\int_S \vec{F} \cdot d\vec{S} + \int\int_{S'} \vec{F} \cdot d\vec{S}.$$

Evaluating the two integrals that are not the one we are really interested in, we see

$$\int\int\int_E div(\vec{F})\, dV = \int\int\int_E 2z\, dV = \int_0^{2\pi} \int_0^1 \int_r^1 2z\, r\, dz\, dr\, d\theta = \frac{\pi}{2}$$

$$\int\int_{S'} \vec{F} \cdot d\vec{S} = \int\int_{S'} \vec{F} \cdot \hat{n}\, dS =$$

$$\int\int_{S'} \left\langle x + \tan^{-1}(y^2),\ -y + \sec(x + z),\ z^2 \right\rangle \cdot \langle 0, 0, 1\rangle\ dS =$$

$$\int\int_{S'} z^2\, dS = \int\int_{S'} 1\, dS = area(S) = \pi$$

Now we see that

$$\frac{\pi}{2} = \int\int_S \vec{F} \cdot d\vec{S} + \pi \qquad \int\int_S \vec{F} \cdot d\vec{S} = \boxed{-\frac{\pi}{2}}$$

6. Let C be the curve consisting of the line segments from $(0,0)$ to $(1,1)$ to $(0,1)$ and back to $(0,0)$. Find the value of

$$\int_C xy\,dx + \sqrt{y^2+1}\,dy.$$

SOLUTION: Whenever you want to integrate a vector field in $\mathbb{R}^2$ around a closed curve, and it looks like the computation might be messy, think of applying Green's Theorem. The curve C is the positively-oriented (draw a picture to check this) boundary of the triangle D with the three corners given, or $0 \le x \le 1$, $x \le y \le 1$. Green's Theorem tells us

$$\int_C P\,dx + Q\,dy = \int\int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

where

$$P = xy \qquad Q = \sqrt{y^2+1}$$

$$\frac{\partial P}{\partial y} = x \qquad \frac{\partial Q}{\partial x} = 0$$

Therefore we have

$$\int_C P\,dx + Q\,dy = \int\int_D -x\,dA = \int_0^1 \int_x^1 -x\,dy\,dx =$$

$$\int_0^1 -x + x^2\,dx = \boxed{-\frac{1}{6}}$$

7. Let $\vec{F}(x,y) = (e^x \sin y + 3y, e^x \cos y + 2x - 2y)$ and $\phi(x,y) = e^x \sin y + 2xy - y^2$.

   (a) Find $\nabla\phi(x,y)$.

   SOLUTION:

   $$\nabla\phi(x,y) = (e^x \sin y + 2y, e^x \cos y + 2x - 2y)$$

   (b) Compute

   $$\int_C \vec{F} \cdot d\vec{r},$$

where C is the positively oriented ellipse $4x^2 + y^2 = 4$. (Hint: make use of part (a) by comparing $\vec{F}$ and $\nabla\phi$.)

SOLUTION: Making use of the hint, we notice

$$\vec{F}(x,y) = \nabla\phi(x,y) + (y,0)$$

and therefore

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla\phi\cdot d\vec{r} + \int_C (y,0)\cdot d\vec{r}$$

By the Fundamental Theorem of Line Integrals, the integral of $\nabla\phi$ around a closed curve is zero, and therefore

$$\int_C \vec{F}\cdot d\vec{r} = \int_C (y,0)\cdot d\vec{r}$$

This integral can be evaluated directly or using Green's Theorem; we will use Green's Theorem, letting D be the region enclosed by C:

$$\int_C (y,0)\cdot d\vec{r} = \int\int_D \frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(y)\,dA = \int\int_D (-1)\,dA =$$

$$-area(D) = -(2)(1)\pi = \boxed{-2\pi}$$

An alternative solution is to apply Green's Theorem directly to the original line integral $\int_C \vec{F}\cdot d\vec{r}$.

8. Evaluate the line integral of the function

$$F(x,y,z) = \left\langle x^2y^3,\, e^{xy+z},\, x+z^2\right\rangle$$

around the circle $x^2+z^2=1$ in the plane $y=0$, oriented counterclockwise as viewed from the positive y-direction.

SOLUTION: Whenever you want to integrate a vector field in $\mathbb{R}^3$ around a closed curve, and it looks like the computation might be messy, think of applying Stokes' Theorem. The circle C in question is the positively-oriented boundary of the disc S given by $x^2+z^2 \le 1$, $y=0$, with the unit normal vector $\hat{n}$ pointing in the positive y direction. That is, $\hat{n} = \hat{j} = \langle 0,1,0\rangle$.

Stokes' Theorem tells us that

$$\int_C \vec{F} \cdot d\vec{r} = \int \int_S curl(\vec{F}) \cdot d\vec{S} = \int \int_S curl(\vec{F}) \cdot \hat{n}\, dS$$

Evaluating the curl of $\vec{F}$ we see

$$curl(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^3 & e^{xy+z} & x+z^2 \end{vmatrix} = \left\langle -e^{xy+z},\ -1,\ ye^{xy+z} - 3x^2y^2 \right\rangle$$

$$curl(\vec{F}) \cdot \hat{n} = \left\langle -e^{xy+z},\ -1,\ ye^{xy+z} - 3x^2y^2 \right\rangle \cdot \langle 0,1,0 \rangle = -1$$

$$\int_C \vec{F} \cdot d\vec{r} = \int \int_S curl(\vec{F}) \cdot \hat{n}\, dS = \int \int_S -1\, dS = -area(S) = \boxed{-\pi}$$

9. Compute the flux of the vector field

$$\vec{F}(x,y,z) = \langle 2x, y, 3z \rangle$$

outward through the sphere of radius 36 centered at the point $(1,2,-1)$.

SOLUTION: The flux of $\vec{F}$ through the surface S is given by the surface integral

$$\int \int_S \vec{F} \cdot d\vec{S}$$

where S is oriented so the normal vector points in the direction of the flux being computed, in our case, outward from the center of the sphere.

Whenever you need to compute the surface integral of a vector field in $\mathbb{R}^3$ over a closed surface, and it looks like the computation might be messy, think of applying Gauss's Theorem (aka the Divergence Theorem). Gauss's Theorem tells us that if S is the positively oriented boundary of the solid region E (that is, the normal vector of S points outward from E) then

$$\int \int_S \vec{F} \cdot d\vec{S} = \int \int \int_E div(\vec{F})\, dV.$$

In our example, E is the solid sphere of radius 36 centered at the point $(1, 2, -1)$,

$$div(\vec{F}) = 2 + 1 + 3 = 6$$

and so

$$\int\int_S \vec{F} \cdot d\vec{S} = \int\int\int_E div(\vec{F})\, dV = \int\int\int_E 6\, dV =$$

$$6(volume(E)) = 6\frac{4\pi}{3}(36)^3 = \boxed{8 \cdot (36)^3\, \pi}.$$

10. Let R be the region in the xy-plane above the x-axis and below the curve C parametrized by $\vec{r}(t) = \langle 1 + t^3, t - t^2 \rangle$ for $t \in [0, 1]$.

   (a) Sketch the region R. (Just do the best you can.)

   SOLUTION: Plugging in $t = 0$ and $t = 1$ we see that C goes from $(1, 0)$ to $(2, 0)$. Also, for $0 \leq t \leq 1$, we have $t - t^2 = t(1 - t) \geq 0$, so C lies in the region $y \geq 0$, on and above the x-axis. Finally, as t goes from 0 to 1, we see that on C we have $x = 1 + t^3$ increases from 1 to 2, and $y = t(1 - t)$ begins at $y = 0$, increases (to a maximum value of $\frac{1}{4}$ at $t = \frac{1}{2}$, if it matters), and then decreases again to $y = 0$. Therefore a point moving along the curve C starts at $(1, 0)$, moves upward and to the right and then downward and to the right, and ends at $(2, 0)$. This should allow you to draw a rough sketch.

   (b) Use Green's Theorem to express the area of R as a line integral.

   SOLUTION: The positively oriented boundary of the region R consists of two pieces, the curve $-C$ (which denotes C with the opposite orientation) and the line segment C' from $(1, 0)$ to $(2, 0)$. We can apply Green's Theorem using the function

$$\vec{F}(x, y) = (P, Q) = (-y, 0)$$

   which satisfies

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

   to get

$$area(R) = \int\int_R 1\, dA = \int\int_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA =$$

$$\int_{-C} \vec{F} \cdot d\vec{r} + \int_{C'} \vec{F} \cdot d\vec{r}$$

We can go a little further by noting that on C' we have $y = 0$ and so $\vec{F} = (-y, 0) = (0, 0)$ and

$$\int_{C'} \vec{F} \cdot d\vec{r} = 0$$

and so we have

$$area(R) = \int_{-C} \vec{F} \cdot d\vec{r} = \boxed{-\int_{C} \vec{F} \cdot d\vec{r}}$$

(c) Compute the area of R by evaluating your line integral from part (b).

SOLUTION:

$$-\int_{C} \vec{F} \cdot d\vec{r} = -\int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,dt =$$

$$-\int_0^1 (-(t-t^2), 0) \cdot (3t^2, 1-2t)\,dt = \int_0^1 3t^3 - 3t^4\,dt = \boxed{\frac{3}{20}}$$

11. Consider the vector field $\vec{F}(x, y, z) = \langle y + z, x - z, zy \rangle$.

(a) Is $\vec{F}$ conservative? Why not?

SOLUTION: No. We know because

$$curl(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & x-z & zy \end{vmatrix} = \langle z+1,\ 1,\ 0 \rangle \neq \vec{0}.$$

(b) Let C be any positively oriented simple closed curve in the xy-plane. Show that $\int_C \vec{F} \cdot d\vec{r} = 0$. (Hint: treat the region D in the xy-plane bounded by C as a surface and apply Stokes's Theorem.)

SOLUTION: Following the hint, the curve C bounds a surface D contained in the xy-plane,, that is, the plane $z = 0$. Because D

is contained in the xy-plane its unit normal vector must be either $\langle 0,0,1\rangle$ or $\langle 0,0,-1\rangle$; because C is oriented so it goes counterclockwise as viewed from the positive z-direction, in order to make C the positively-oriented boundary of D, we must orient D so that $\hat{n} = \langle 0,0,1\rangle$. Now, by Stokes' Theorem, we have

$$\int_C \vec{F}\cdot d\vec{r} = \int\int_D curl(\vec{F})\cdot d\vec{S} = \int\int_D curl(\vec{F})\cdot \hat{n} dS =$$

$$\int\int_D \langle z+1,\ 1,\ 0\rangle \cdot \langle 0,0,1\rangle\ dS = \int\int_S 0\, dS = 0.$$

12. Show that if $\vec{F} = (F_1,\ F_2)$ is a vector field on $\mathbb{R}^2$ such that, on all of $\mathbb{R}^2$, the component functions F_1 and F_2 have continuous partial derivatives and

$$\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 0,$$

then the flux integral

$$\int_C \vec{F}\cdot \hat{N}\, ds$$

is path-independent. That is, if C_1 and C_2 are two piecewise smooth curves with the same endpoints, then

$$\int_{C_1} \vec{F}\cdot \hat{N}\, ds = \int_{C_2} \vec{F}\cdot \hat{N}\, ds.$$

If it helps, you may assume C_1 and C_2 do not cross, or have any points in common except their endpoints.

In this problem $\hat{N}$ denotes the unit normal vector to the curve. Hint: Use the second version of Green's Theorem; see page 1103 of the textbook.

SOLUTION:

There are (at least) two possible proofs you can give here. We outline the two methods.

Method I: Assume C_1 and C_2 do not cross, or have any points in common except their endpoints, and use the hint. The region D between C_1 and C_2 has as its positively oriented boundary C_1 together with

C_2, one of them with the reversed orientation. Let us say the boundary of D is $C_1 + (-C_2)$. Then, applying the second version of Green's Theorem, we see

$$\int\int_D \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\, dA = \int_{C_1} \vec{F} \cdot \hat{N}\, ds + \int_{-C_2} \vec{F} \cdot \hat{N}\, ds$$

$$\int\int_D 0\, dA = \int_{C_1} \vec{F} \cdot \hat{N}\, ds - \int_{C_2} \vec{F} \cdot \hat{N}\, ds$$

$$\int_{C_1} \vec{F} \cdot \hat{N}\, ds - \int_{C_2} \vec{F} \cdot \hat{N}\, ds = 0$$

$$\int_{C_1} \vec{F} \cdot \hat{N}\, ds = \int_{C_2} \vec{F} \cdot \hat{N}\, ds$$

Method 2: Rewrite

$$\int_C \vec{F} \cdot \hat{N}\, ds = \int_C F_1\, dy - F_2\, dx = \int_C (-F_2)\, dx + (F_2)\, dy = \int_C \vec{G} \cdot d\vec{r}$$

where

$$\vec{G} = (-F_2, F_1) = (G_1, G_2)$$

Now, on all of $\mathbb{R}^2$ we have

$$\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = \frac{\partial F_1}{\partial x} - \frac{\partial(-F_2)}{\partial y} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 0$$

and because $\mathbb{R}^2$ is a simply connected region, this tells us that $\vec{G}$ is conservative and line integrals of $\vec{G}$ are path-independent. Therefore we have

$$\int_{C_1} \vec{F} \cdot \hat{N}\, ds = \int_{C_1} \vec{G} \cdot d\vec{r} = \int_{C_2} \vec{G} \cdot d\vec{r} \int_{C_2} \vec{F} \cdot \hat{N}\, ds$$