Math 128: Lecture 9

April 14, 2014

1. The adjoint action of $\mathfrak{g}_\alpha$ sends $\mathfrak{g}_\beta$ to $\mathfrak{g}_{\alpha+\beta}$.
2. If $x_\alpha \in g_\alpha$ $(\alpha \neq 0)$, then x_α is nilpotent.
3. If $\alpha \neq -\beta$, then $\langle \mathfrak{g}_\alpha, \mathfrak{g}_\beta \rangle = 0$.
4. (Symmetry) If $\alpha \in R$, then $-\alpha \in R$.
5. The set $\{h_\alpha \mid \alpha \in R\}$ spans $\mathfrak{h}$, and so R spans $\mathfrak{h}^*$.
6. If $x_\alpha \in \mathfrak{g}_\alpha$ and $y_\alpha \in \mathfrak{g}_{-\alpha}$ then $[x_\alpha, y_\alpha] = \langle x_\alpha, y_\alpha \rangle h_\alpha$. Further, there is some y_α for which $\langle x_\alpha, y_\alpha \rangle \neq 0$, so $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C} h_\alpha$.
7. For all $\alpha \in R$, $\langle h_\alpha, h_\alpha \rangle \neq 0$.
8. Every non-zero $x_\alpha \in \mathfrak{g}_\alpha$ is part of an $\mathfrak{sl}_2$-triple,

$$\mathfrak{s}_\alpha = \langle x_\alpha, y_\alpha, h_{\alpha^\vee} \rangle, \quad \text{with} \quad y_\alpha \in \mathfrak{g}_{-\alpha} \text{ and } h_{\alpha^\vee} = \frac{2h_\alpha}{\langle h_\alpha, h_\alpha \rangle}.$$

9. If $\alpha \in R$ and $c\alpha \in R$ for some $c \in \mathbb{C}^\times$, then $c = \pm 1$.
10. For $\alpha \neq 0$, $\mathfrak{g}_\alpha = 0$ or $\mathfrak{g}_\alpha$ is one-dimensional. So if $\langle , \rangle$ is the Killing form, then for any $h_1, h_2 \in \mathfrak{h}$,

$$\langle h_1, h_2 \rangle = \sum_{\alpha \in R} \alpha(h_1)\alpha(h_2).$$

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11. For $\alpha, \beta \in R$,
    (a) $\beta(h_{\alpha^\vee}) \in \mathbb{Z}$,
    (b) $\beta - \beta(h_{\alpha^\vee})\alpha \in R$, and
    (c) if $\beta \neq \pm\alpha$, and a and b are the largest non-negative integers such that

    $$\beta - a\alpha \in R \qquad \text{and } \beta + b\alpha \in R,$$

    then $\beta + i\alpha \in R$ for all $-a \leq i \leq b$ and $\beta(h_{\alpha^\vee}) = a - b$.

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12. (Rationality) Let $B \subseteq R$ be a base for R.
    (a) $R \subseteq \mathbb{Q}B$.
    (b) For any $\alpha, \beta \in R$, $\langle \alpha, \beta \rangle \in \mathbb{Q}$.
    (c) The restriction of $\langle , \rangle$ to $\mathfrak{h}^*_{\mathbb{Q}} = \mathbb{Q}B$ and $\mathfrak{h}^*_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Q}} \mathfrak{h}^*_{\mathbb{Q}}$ is positive definite (so that $\mathfrak{h}^*_{\mathbb{Q}}$, $\mathfrak{h}^*_{\mathbb{R}}$, $\mathfrak{h}_{\mathbb{Q}}$, and $\mathfrak{h}_{\mathbb{R}}$ are all Euclidean spaces with inner product $\langle , \rangle$).

The Weyl group

Let $\mathfrak{h}_\alpha$ be the hyperplane in the real Euclidean space $\mathfrak{h}^*_{\mathbb{R}}$ given by

$$\mathfrak{h}_\alpha = \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle \lambda, \alpha \rangle = 0\}.$$

(Notice that $\mathfrak{h}_\alpha = \mathfrak{h}_{-\alpha}$.)

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(Notice that $\mathfrak{h}_\alpha = \mathfrak{h}_{-\alpha}$.)

Then σ_α extends to a map on $\mathfrak{h}^*_{\mathbb{R}}$, given by

$$\begin{aligned} \sigma_\alpha : & \mathfrak{h}^*_{\mathbb{R}} \to \mathfrak{h}^*_{\mathbb{R}} \\ & \lambda \mapsto \lambda - 2\frac{\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle}\alpha, \end{aligned}$$

which geometrically reflects weights across the hyperplane $\mathfrak{h}_\alpha$.

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The group W generated by $\{\sigma_\alpha \mid \alpha \in R^+\}$ is called the *Weyl group* associated to $\mathfrak{g}$.