Math 128: Lecture 7

April 7, 2014

Quick comment on bases of the classicals

Answers to Exercise 1 are up.

Basis of A_r:

$$\{E_{ii} - E_{i+1,i+1} \mid i = 1, \ldots, r\} \sqcup \{E_{ij}, E_{ij} \mid 1 \leq i < j \leq r+1\}.$$

Basis of B_r:

$$\begin{aligned}\{E_{i+1,j+1} - E_{j+1+r,i+1+r} \mid 1 \leq i, j \leq n\} \\ \sqcup \{E_{i+1,r+j} - E_{j,r+i}, E_{r+i,j} - E_{r+j,i} \mid 1 \leq i < j \leq r\} \\ \sqcup \{E_{1,r+i+1} - E_{i+1,1}, E_{1,i+1} - E_{r+i+1,1} \mid i = 1, \ldots, r\}.\end{aligned}$$

Basis of C_r:

$$\{E_{ij} - E_{j+r,i+r} \mid 1 \leq i, j \leq r\} \sqcup \{E_{i,r+j} + E_{j,r+i}, E_{r+i,j} + E_{r+j,i} \mid 1 \leq i \leq j \leq r\}.$$

Basis of D_r:

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Advantage: Way easier to calculate.
Disadvantage: What's the Cartan?

Last time:

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The *weights* of a Cartan $\mathfrak{h}$ is the dual set $\mathfrak{h}^* = \{\mu : \mathfrak{h} \to \mathbb{C}\}$.
The restriction of the Killing form to $\mathfrak{h}$ is non-degenerate, and so the map

$$\mathfrak{h} \longrightarrow \mathfrak{h}^* \quad \text{defined by} \quad h \mapsto \langle h, \cdot \rangle$$

is an isomorphism. Let h_μ be the unique element of $\mathfrak{h}$ such that

$$\langle h_\mu, h \rangle = \mu(h) \qquad \text{for all } h \in \mathfrak{h}.$$

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With $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid \mathrm{ad}_h(x) = \alpha(h)x\}$, the set of weights

$$R = \{\alpha \in \mathfrak{h}^* \mid \alpha \neq 0, \mathfrak{g}_\alpha \neq 0\}$$

is called the set of *roots* of $\mathfrak{g}$.

Some facts about roots

For $\alpha \in R$, $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid \mathrm{ad}_h(x) = \alpha(h)x\} \neq 0$.
Let $\langle , \rangle$ be the Killing form. Recall, invariant means $\langle [x, y], z\rangle = -\langle y, [x, z]\rangle$.

1. The adjoint action of $\mathfrak{g}_\alpha$ sends $\mathfrak{g}_\beta$ to $\mathfrak{g}_{\alpha+\beta}$:

   $$\text{for } x_\alpha \in \mathfrak{g}_\alpha, \qquad \mathrm{ad}_x : \mathfrak{g}_\alpha \to \mathfrak{g}_{\alpha+\beta}.$$

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7. For all $\alpha \in R$, $\langle h_\alpha, h_\alpha\rangle \neq 0$.
8. Every non-zero $x_\alpha \in \mathfrak{g}_\alpha$ is part of an $\mathfrak{sl}_2$-triple,

   $$\mathfrak{s}_\alpha = \langle x_\alpha, y_\alpha, h_{\alpha^\vee}\rangle, \quad \text{with} \quad y_\alpha \in \mathfrak{g}_{-\alpha} \text{ and } h_{\alpha^\vee} = \frac{2h_\alpha}{\langle h_\alpha, h_\alpha\rangle}.$$