Math 128: Lecture 6

April 4, 2014

From last time:

Let U be a Hopf algebra with module M.

- A *bilinear form* is a map $\langle , \rangle : M \otimes M \to \mathbb{C}$.
- A bilinear form is *symmetric* if $\langle m, n \rangle = \langle n, m \rangle$ for all $x, y \in M$.
- A bilinear form is *invariant* if $\langle xm, n \rangle = \langle m, S(x)n \rangle$ for all $x \in U$.
- A bilinear form is *nondegenerate* if $\langle x, M \rangle \neq 0$ for all $x \in M$.

The *Killing form* on a Lie algebra $\mathfrak{g}$ is

$$\langle x, y \rangle = \mathrm{Tr}(\mathrm{ad}_x \mathrm{ad}_y),$$

and is invariant, bilinear, and symmetric. If $\mathfrak{g}$ is semisimple, it is also nondegenerate.

If $\mathfrak{g}$ is simple, then every nondegenerate invariant bilinear symmetric (NIBS) form is a constant multiple of the Killing form.

Jordan canonical form: Every linear map can be decomposed into blocks,

$$\begin{pmatrix} \lambda & 1 & & & 0 \\ & \lambda & 1 & & \\ & & \ddots & & \\ & & & & 1 \\ 0 & & & & \lambda \end{pmatrix}$$

Jordan canonical form: Every linear map can be decomposed into blocks,

$$\begin{pmatrix} \lambda & 1 & & & 0 \\ & \lambda & 1 & & \\ & & \ddots & & \\ & & & & 1 \\ 0 & & & & \lambda \end{pmatrix} = \underbrace{\lambda I}_{s} + \underbrace{\begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & & \\ & & & & 1 \\ 0 & & & & 0 \end{pmatrix}}_{n}$$

which are the sum of a *semisimple* (diagonalizable) part s and a *nilpotent* ($n^\ell = 0$ for some ℓ) part n.

Jordan canonical form: Every linear map can be decomposed into blocks,

$$\begin{pmatrix} \lambda & 1 & & & 0 \\ & \lambda & 1 & & \\ & & \ddots & & \\ & & & & 1 \\ 0 & & & & \lambda \end{pmatrix} = \underbrace{\lambda I}_{s} + \underbrace{\begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & & \\ & & & & 1 \\ 0 & & & & 0 \end{pmatrix}}_{n}$$

which are the sum of a *semisimple* (diagonalizable) part s and a *nilpotent* ($n^\ell = 0$ for some ℓ) part n. Further, $s = \lambda I$ is central, so $sn = ns$.

Jordan canonical form: Every linear map can be decomposed into blocks,

$$\begin{pmatrix} \lambda & 1 & & & 0 \\ & \lambda & 1 & & \\ & & \ddots & & \\ & & & & 1 \\ 0 & & & & \lambda \end{pmatrix} = \underbrace{\lambda I}_{s} + \underbrace{\begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & & \\ & & & & 1 \\ 0 & & & & 0 \end{pmatrix}}_{n}$$

which are the sum of a *semisimple* (diagonalizable) part s and a *nilpotent* ($n^\ell = 0$ for some ℓ) part n. Further, $s = \lambda I$ is central, so $sn = ns$.

So every linear transformation x can be expressed as

$$x = x_s + x_n \quad \text{with} \quad \begin{array}{l} x_s \text{ semisimple,} \\ x_n \text{ nilpotent, and} \\ x_n x_s = x_s x_n \end{array}$$

Let $\mathfrak{g}$ be a FDSSC Lie algebra. For $x \in \mathfrak{g}$, x is semisimple/nilpotent if ad_x is semisimple/nilpotent.

Theorem (Jordan-Chevalley decomposition)

For $x \in \mathfrak{g}$, then there exist unique x_s semisimple and x_n nilpotent satisfying

$$x = x_s + x_n \quad \textit{and} \quad [x_s, x_n] = 0.$$

Further if $y \in \mathfrak{g}$ satisfies $[x, y] = 0$, then $[x_s, y] = [x_n, y] = 0$.

(see, for example, [Hum, §4.2] or [Ser, §I.5])

Let $\mathfrak{g}$ be a FDSSC Lie algebra. For $x \in \mathfrak{g}$, x is semisimple/nilpotent if ad_x is semisimple/nilpotent.

Theorem (Jordan-Chevalley decomposition)

For $x \in \mathfrak{g}$, then there exist unique x_s semisimple and x_n nilpotent satisfying

$$x = x_s + x_n \quad \textit{and} \quad [x_s, x_n] = 0.$$

Further if $y \in \mathfrak{g}$ satisfies $[x, y] = 0$, then $[x_s, y] = [x_n, y] = 0$.

(see, for example, [Hum, §4.2] or [Ser, §I.5])

Theorem (Jasobson-Morozov)

If x is a nilpotent element of a finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$, then there exist nilpotent y and semisimple h in $\mathfrak{g}$ such that

$$[x, y] = h, \qquad [h, x] = 2x, \qquad [h, y] = -2y.$$

This choice is relatively unique (with some changes in constants). We call $\{x, y, h\}$ an $\mathfrak{sl}_2$ triple.

A *Cartan subalgebra* of $\mathfrak{g}$ is a maximal abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$ consisting of semisimple elements.

A *Cartan subalgebra* of $\mathfrak{g}$ is a maximal abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$ consisting of semisimple elements.

Some facts about Cartans: (see for example [Ser, Ch. III])

1. Cartan subalgebras are generated by taking a (nice) semisimple element h and setting

$$\mathfrak{h} = \{g \in \mathfrak{g} \mid \mathrm{ad}_h(g) = 0\}$$

A *Cartan subalgebra* of $\mathfrak{g}$ is a maximal abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$ consisting of semisimple elements.

Some facts about Cartans: (see for example [Ser, Ch. III])

1. Cartan subalgebras are generated by taking a (nice) semisimple element h and setting
$$\mathfrak{h} = \{g \in \mathfrak{g} \mid \mathrm{ad}_h(g) = 0\}$$
2. Cartan subalgebras exist and are unique up to inner automorphisms.

A *Cartan subalgebra* of $\mathfrak{g}$ is a maximal abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$ consisting of semisimple elements.

Some facts about Cartans: (see for example [Ser, Ch. III])

1. Cartan subalgebras are generated by taking a (nice) semisimple element h and setting
$$\mathfrak{h} = \{g \in \mathfrak{g} \mid \mathrm{ad}_h(g) = 0\}$$
2. Cartan subalgebras exist and are unique up to inner automorphisms.
3. The centralizer of $\mathfrak{h}$ is $\mathfrak{h}$.

A *Cartan subalgebra* of $\mathfrak{g}$ is a maximal abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$ consisting of semisimple elements.

Some facts about Cartans: (see for example [Ser, Ch. III])

1. Cartan subalgebras are generated by taking a (nice) semisimple element h and setting
$$\mathfrak{h} = \{g \in \mathfrak{g} \mid \mathrm{ad}_h(g) = 0\}$$
2. Cartan subalgebras exist and are unique up to inner automorphisms.
3. The centralizer of $\mathfrak{h}$ is $\mathfrak{h}$.
4. All elements of $\mathfrak{h}$ are semisimple.

A *Cartan subalgebra* of $\mathfrak{g}$ is a maximal abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$ consisting of semisimple elements.

Some facts about Cartans: (see for example [Ser, Ch. III])

1. Cartan subalgebras are generated by taking a (nice) semisimple element h and setting
$$\mathfrak{h} = \{g \in \mathfrak{g} \mid \mathrm{ad}_h(g) = 0\}$$
2. Cartan subalgebras exist and are unique up to inner automorphisms.
3. The centralizer of $\mathfrak{h}$ is $\mathfrak{h}$.
4. All elements of $\mathfrak{h}$ are semisimple.
5. The restriction of the Killing form to $\mathfrak{h}$ is non-degenerate.

A *Cartan subalgebra* of $\mathfrak{g}$ is a maximal abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$ consisting of semisimple elements.

Some facts about Cartans: (see for example [Ser, Ch. III])

1. Cartan subalgebras are generated by taking a (nice) semisimple element h and setting
$$\mathfrak{h} = \{g \in \mathfrak{g} \mid \mathrm{ad}_h(g) = 0\}$$
2. Cartan subalgebras exist and are unique up to inner automorphisms.
3. The centralizer of $\mathfrak{h}$ is $\mathfrak{h}$.
4. All elements of $\mathfrak{h}$ are semisimple.
5. The restriction of the Killing form to $\mathfrak{h}$ is non-degenerate.

The *rank* of a semisimple Lie algebra is defined by

$$\mathrm{rank}(\mathfrak{g}) = \dim(\mathfrak{h}).$$

The *weights* of a Cartan $\mathfrak{h}$ is the dual set $\mathfrak{h}^* = \{\mu : \mathfrak{h} \to \mathbb{C}\}$.

The *weights* of a Cartan $\mathfrak{h}$ is the dual set $\mathfrak{h}^* = \{\mu : \mathfrak{h} \to \mathbb{C}\}$.

Let $\mathfrak{g}_\alpha = \{g \in \mathfrak{g} \mid \mathrm{ad}_h(x) = \alpha(h)x\}$. The set of weights

$$R = \{\alpha \in \mathfrak{h}^* \mid \alpha \neq 0, g_\alpha \neq 0\}$$

is called the set of *roots* of $\mathfrak{g}$.

The *weights* of a Cartan $\mathfrak{h}$ is the dual set $\mathfrak{h}^* = \{\mu : \mathfrak{h} \to \mathbb{C}\}$.

Let $\mathfrak{g}_\alpha = \{g \in \mathfrak{g} \mid \mathrm{ad}_h(x) = \alpha(h)x\}$. The set of weights

$$R = \{\alpha \in \mathfrak{h}^* \mid \alpha \neq 0, g_\alpha \neq 0\}$$

is called the set of *roots* of $\mathfrak{g}$.

Let $\langle , \rangle$ be a NIBS form on $\mathfrak{g}$. Then the map

$$\begin{array}{ccc} \mathfrak{h} & \longrightarrow & \mathfrak{h}^* \\ h & \mapsto & \langle h, \cdot \rangle \\ h_\mu & \mapsto & \mu \end{array} \qquad \text{is an isomorphism,}$$

where h_μ is the unique element of $\mathfrak{h}$ such that

$$\langle h_\mu, h \rangle = \mu(h) \qquad \text{for all } h \in \mathfrak{h}.$$

The *weights* of a Cartan $\mathfrak{h}$ is the dual set $\mathfrak{h}^* = \{\mu : \mathfrak{h} \to \mathbb{C}\}$.

Let $\mathfrak{g}_\alpha = \{g \in \mathfrak{g} \mid \mathrm{ad}_h(x) = \alpha(h)x\}$. The set of weights

$$R = \{\alpha \in \mathfrak{h}^* \mid \alpha \neq 0, g_\alpha \neq 0\}$$

is called the set of *roots* of $\mathfrak{g}$.

Let $\langle , \rangle$ be a NIBS form on $\mathfrak{g}$. Then the map

$$\begin{array}{ccc} \mathfrak{h} & \longrightarrow & \mathfrak{h}^* \\ h & \mapsto & \langle h, \cdot \rangle \\ h_\mu & \mapsto & \mu \end{array} \qquad \text{is an isomorphism,}$$

where h_μ is the unique element of $\mathfrak{h}$ such that

$$\langle h_\mu, h \rangle = \mu(h) \qquad \text{for all } h \in \mathfrak{h}.$$

Define $\langle , \rangle : \mathfrak{h}^* \otimes \mathfrak{h}^* \to \mathbb{C}$ by $\langle \mu, \lambda \rangle = \langle h_\mu, h_\lambda \rangle$.