Math 128: Lecture 4

March 31, 2014

Recall if $\mathfrak{g}$ is a Lie algebra with modules M and N, then $x \in \mathfrak{g}$ acts on $m \otimes n \in M \otimes N$ by

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In general, if $w = x_1 x_2 \ldots x_\ell \in U\mathfrak{g}$ with $x_i \in \mathfrak{g}$, then

$$\Delta(w) = \Delta(x_1)\Delta(x_2) \cdots \Delta(x_\ell).$$

Representations of $\mathfrak{sl}_2(\mathbb{C})$

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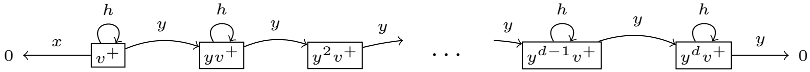
So far:



In summary, the $\mathfrak{sl}_2$-action is given by:

- h is a diagonal matrix with μ, $\mu - 2$, $\mu - 4$, $\ldots$, $\mu - 2d$ on the diagonal,
- y has 1's on the sub-diagonal and zeros elsewhere, and
- x has the weights μ, $2\mu - 2$, $3\mu - 6$, $\ldots$, $d(\mu - (d-1))$ on the super-diagonal.

$$h = \begin{pmatrix} \mu & & & & \\ & \mu - 2 & & & \\ & & \mu - 4 & & \\ & & & \ddots & \\ & & & & \mu - 2d \end{pmatrix}$$

$$y = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & & \ddots & \\ & & & 1 & 0 \end{pmatrix} \quad x = \begin{pmatrix} 0 & \mu & & & \\ & 0 & 2\mu - 2 & & \\ & & 0 & 3\mu - 6 & \\ & & & \ddots & d(\mu - (d-1)) \\ & & & & 0 \end{pmatrix}$$

Theorem

The simple finite dimensional $\mathfrak{sl}_2$ *modules* $L(d)$ *are indexed by* $d \in \mathbb{Z}_{\geq 0}$ *with basis* $\{v^+, yv^+, y^2v^+, \ldots, y^dv^+\}$ *and action*

$$\begin{aligned}
h(y^\ell v^+) &= (d-2\ell)(y^\ell v^+), \\
x(y^\ell v^+) &= \ell(d+1-\ell)(y^{\ell-1}v^+), \qquad \textit{with } xv^+ = 0 \textit{ and} \\
y(y^\ell v^+) &= y^{\ell+1}v^+, \qquad \textit{with } y^{d+1}v^+ = 0.
\end{aligned}$$

Identifying finite dimensional $\mathfrak{sl}_2$-modules

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Let M be a (not nec. simple) $\mathfrak{sl}_2$-module.
As a $\mathbb{C}h$-module,

$$M = \bigoplus_{\mu \in \mathbb{C}} M_\mu \quad \text{where} \quad M_\mu = \{m \in M \mid hm = \mu m\},$$

is the μ *weight space*.
(Remember, “weight”=“eigen. . .”)