Math 128: Lecture 24

May 21, 2014

Diagram algebras

A diagram algebra has vector space the linear span of (some class of) diagrams on $2k$ vertices, which can be

1. graphs with certain conditions;
2. decorated graphs with certain conditions;
3. braids, sometimes with punctures;
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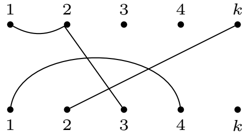

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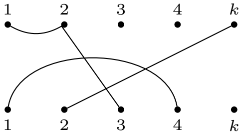
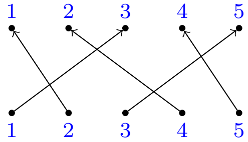


The multiplication is given by concatenation, with rules for resolving new artifacts arriving in the diagrams.

Examples of diagram algebras

Our favorite examples encode endomorphisms of a tensor space that commute with the action of another algebra.

Example 1: The symmetric group S_k as diagrams:



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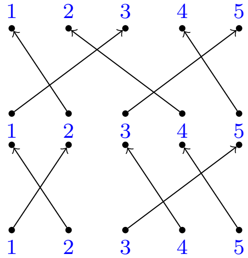

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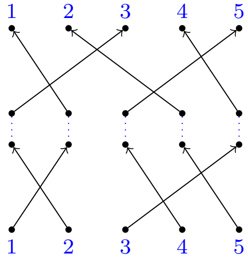

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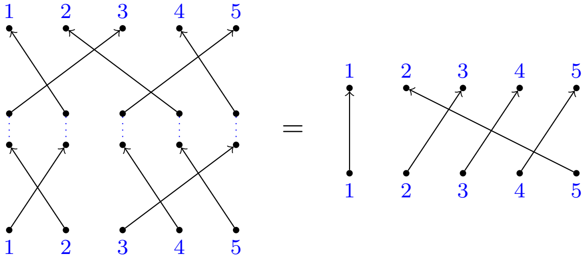

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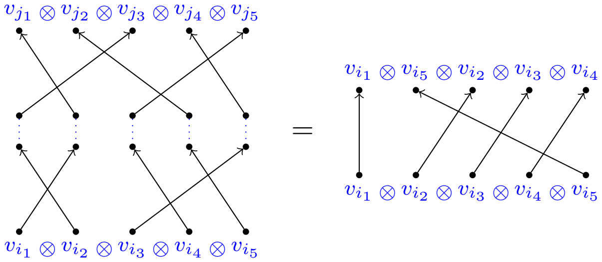

These diagrams encode endomorphisms of $L(\square)^{\otimes k}$ that commute with the action of $U\mathfrak{sl}_n$, $U\mathfrak{gl}_n$, $\mathbb{C}\mathrm{SL}_n$, and $\mathbb{C}\mathrm{GL}_n$.

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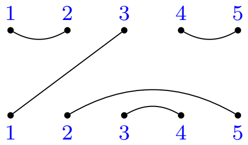

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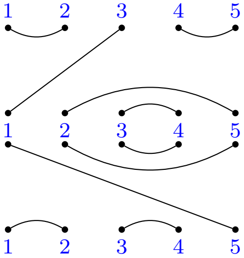

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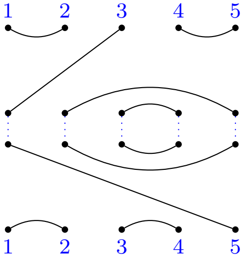

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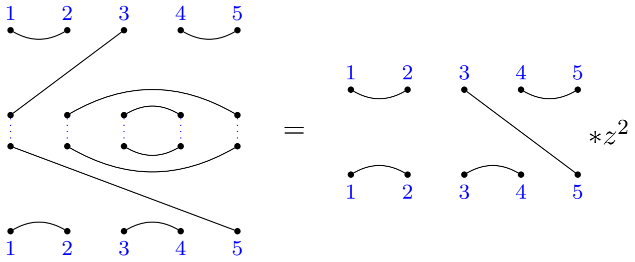

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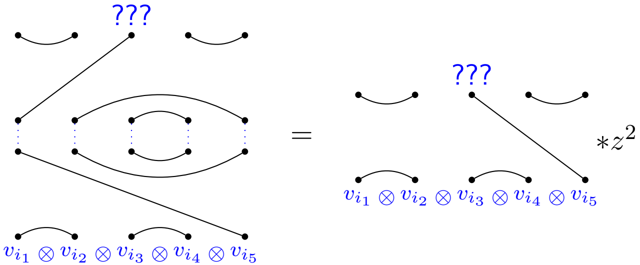

When $z = 2$, these diagrams encode endomorphisms of $L(\square)^{\otimes k}$ that commute with the action of $U\mathfrak{sl}_2$, $U\mathfrak{gl}_2$, $\mathbb{C}\mathrm{SL}_2$, and $\mathbb{C}\mathrm{GL}_2$.

Last time:

The Temperley-Lieb algebra $TL_k(z)$ is generated over $\mathbb{C}[z^{\pm 1}]$ by $e_1, \dots, e_{k-1}$ with relations

$$e_i^2 = ze_i, \quad e_i e_{i\pm 1} e_i = e_i, \quad e_i e_j = e_j e_i \text{ for } |i-j| > 1.$$

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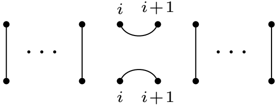
Define the action of $TL_k(2)$ on $L(\square)^{\otimes k}$ via the action of $\mathbb{C}S_k$ by



$$= e_i = 1 - s_i = 2p^{(i)}_{\boxminus}.$$

Then since $TL_k(2) = \mathbb{C}S_k / \langle p^{(i)}_{\boxminus} \mid i = 1, \ldots, k-2 \rangle$, we have TL_k centralizes $U\mathfrak{sl}_2$ in $\mathrm{End}(L(\square)^{\otimes k})$.

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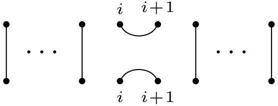

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For $U\mathfrak{sl}_2$, $L(\begin{smallmatrix}\square\\\square\end{smallmatrix}) = L(\emptyset)$, so e_i is really the projection into the trivial component!

Fundamental and dominant integral weights

Type B_r, C_r, and D_r:

$\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$ for $i = 1, \ldots r-1$ (for types B_r, C_r) or $r-2$ (for type D_r)

and

$$P^+ = \{\lambda_1\varepsilon_1 + \cdots + \lambda_r\varepsilon_r \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq |\lambda_r| \geq 0 \text{ and } *\}, \text{ where...}$$

Type B_r: $\omega_r = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_r)$, so

$$* : \quad \lambda_r \geq 0 \text{ and } \lambda_i \in \mathbb{Z} \text{ for all } i \text{ or } \lambda_i \in \mathbb{Z} + \frac{1}{2} \text{ for all } i.$$

So $P^+ = \{$ part'ns with no more than r parts, shifted by 0 or $\frac{1}{2}$ $\}$.

Type C_r: $\omega_r = \varepsilon_1 + \cdots + \varepsilon_r$, so

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Type D_r: $\omega_{r-1} = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{r-1} - \varepsilon_r)$ and $\omega_r = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_r)$, so

$$* : \quad \lambda_i \in \mathbb{Z} \text{ for all } i \text{ or } \lambda_i \in \mathbb{Z} + \frac{1}{2} \text{ for all } i.$$

So $P^+ = \{$ B_r part'ns with last row marked with $\pm$ $\}$.

In all cases B_r, C_r, D_r, when λ isn't too tall

$$L(\lambda) \otimes L(\square) = \bigoplus_{\mu \in \lambda^{\pm}} L(\mu),$$

where

$\lambda^{\pm} = \{$ "partitions" obtained from λ by adding or removing a box $\}$.

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Let

$$s_i : v_{j_i} \otimes v_{j_{i+1}} \to v_{j_{i+1}} \otimes v_{j_i} \qquad \text{and} \qquad e_i = np_\emptyset^{(i)},$$

where $n = 2r + 1$ (in type B_r) or $2r$ (in types C_r or D_r).

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Fix $\epsilon = \pm 1$. The Brauer algebra $B_k(\epsilon, z)$ is generated over $C = \mathbb{C}[z^{\pm 1}]$ by $CS_k = C\langle s_1, \ldots, s_{k-1}\rangle$ and $TL_k(z) = C\langle e_1, \ldots, e_{k-1}\rangle$, with additional relations

$$e_i s_i = s_i e_i = \epsilon e_i, \quad e_i s_j = s_j e_i \text{ for } |i - j| > 1,$$

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$B_k(\epsilon, z)$ centralizes $U\mathfrak{g}$ in $\mathrm{End}(L(\square)^{\otimes k})$ when $z = n$, $\epsilon = \begin{cases} 1 & \mathfrak{g} = B_r, D_r, \\ -1 & \mathfrak{g} = C_r. \end{cases}$

Adjusting the tensor space

Let $\mathfrak{g}$ have basis $\mathcal{B} = \{b_1, \ldots, b_d\}$, and dual basis $\mathcal{B}^* = \{b_1^*, \ldots, b_d^*\}$ with respect to $\langle , \rangle$.

Recall the Casismir element, given by

$$\kappa = \sum_i b_i b_i^* \in U\mathfrak{g}$$

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is independent of choice of basis, and so $\sum_i b_i \otimes b_i^* = \sum_i b_i^* \otimes b_i$.

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is independent of choice of basis, and so $\sum_i b_i \otimes b_i^* = \sum_i b_i^* \otimes b_i$. Namely,

$$\Delta(\kappa) = \sum_i (b_i \otimes 1 + 1 \otimes b_i)(b_i^* \otimes 1 + 1 \otimes b_i^*) = \kappa \otimes 1 + 1 \otimes \kappa + 2\gamma.$$

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Homework:

1. $\kappa \otimes 1$ and $\Delta(\kappa)$ have commuting actions on $M \otimes N$.
2. In type A_r, κ acts on the irreducible module $L(\lambda)$ by the constant

$$\langle \lambda, \lambda + 2\rho \rangle = 2 \sum_{\text{box} \in \lambda} c(\text{box}) + (r+1)|\lambda| + \frac{|\lambda|^2}{r+1},$$

   where the content of a box in a partition λ is

$$c(\text{box}) = \text{row}(\text{box}) - \text{col}(\text{box}).$$

3. Calculate the action of $\Delta(\kappa) - \kappa \otimes 1$ on the $L(\mu)$ in $L(\lambda) \otimes L(\Box)$, giving the answer in terms of the content of the box added.

Adjusting the tensor space - Type A_r [Lu88], [KR02]

The polynomial ring $\mathbb{C}[x_1, x_2, \ldots, x_k] = \mathbb{C}[x]$ has an action by $\mathbb{C}S_k$ by

$$w \cdot (x_1^{c_1} \cdots x_k^{c_k}) = x_1^{c_{w^{-1}(1)}} \cdots x_k^{c_{w^{-1}(k)}};$$

write $x^{\mathbf{c}} = x_1^{c_1} \cdots x_k^{c_k}$ and $wx^{\mathbf{c}} = x^{w\mathbf{c}}$ for short.

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The graded Hecke algebra of type A or the degenerate affine Hecke algebra of type A $\mathbb{H}_k$ is

$$\mathbb{H}_k = \mathbb{C}[x] \otimes \mathbb{C}S_k$$

with additional relations

$$s_i x^{\mathbf{c}} = x^{s_i \mathbf{c}} s_i - \frac{x^{\mathbf{c}} - x^{s_i \mathbf{c}}}{x_i - x_{i+1}}.$$

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This is equivalent to

$$s_i x_i = x_{i+1} s_i - 1 \qquad \text{and} \qquad s_i x_j = x_j s_i \text{ for } j \neq i, i+1.$$

Adjusting the tensor space - Type A_r [DRV1, DRV2]

The graded Hecke algebra of type A is
$\mathbb{H}_k = \mathbb{C}[x] \otimes \mathbb{C}S_k / \langle s_i x_i = x_{i+1} s_i - 1,\ s_i x_j = x_j s_i$ for $j \neq i, i+1 \rangle$.
For $\lambda \in P^+$, $\mathbb{H}_k$ acts on

$$L(\lambda) \otimes L(\square)^{\otimes k} = \mathbb{C}\left\{ u_j \otimes v_{j_1} \otimes \cdots \otimes v_{j_k} \mid \begin{matrix} j = 1, \ldots, \dim(L(\lambda)), \\ j_i = 1, \ldots, r+1 \end{matrix} \right\}$$

by

$$s_i : v_{j_i} \otimes v_{j_{i+1}} \to v_{j_{i+1}} \otimes v_{j_i},$$

as before, and

$$x_i = \Delta(\kappa) \otimes 1 - \kappa \otimes 1 \otimes 1$$

on

$$\left(L(\lambda) \otimes L(\square)^{\otimes i-1}\right) \otimes L(\square) \otimes \left(L(\square)^{\otimes k-i}\right).$$

Adjusting the tensor space - Type B_r [Na96], [DRV1], [DRV2]

The degenerate affine Birman-Murakami-Wenzl algebra $\mathbb{B}_k(\epsilon, z_0, z_1, z_2, \dots)$ is

$$\mathbb{H}_k = \mathbb{C}[x] \otimes B_k(\epsilon, z_0)$$

$$s_i x_i = x_{i+1} s_i - (1 - e_i) \qquad (x_i + x_{i+1}) e_i = e_i (x_i + x_{i+1}) = 0,$$

$$e_1 x_1^\ell e_i = z_\ell e_1,$$

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$\mathbb{B}_k(\epsilon, z_0, z_1, z_2, \dots)$ acts on $L(\lambda) \otimes L(\square)^{\otimes k}$ for appropriate choices of $\epsilon, z_0, z_1, \dots$, and centralizes the action of $U\mathfrak{g}$ for $\mathfrak{g} = B_r$, C_r, or D_r.