# Math 128: Lecture 21

May 14, 2014

# Decomposing modules

Let $A$ be a semisimple algebra over $\mathbb{C}$.
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*Maschke's theorem* says that the decomposition

$$M = \bigoplus_{\lambda \in \widehat{M}} M^{(\lambda)}, \quad \text{where } \widehat{M} = \{\lambda \in \widehat{A} \mid M^{(\lambda)} \neq 0\}, \quad \text{is unique,}$$

whereas

$$M^{(\lambda)} = \bigoplus_{i=1,\ldots,m_M(\lambda)} A^\lambda = m_M(\lambda) A^\lambda \qquad \text{is not unique.}$$

# Finite-dimensional case

If $A$ is finite-dimensional, *Wedderburn's theorem* says

$$A \cong \bigoplus_{\lambda \in \widehat{A}} \mathrm{End}(A^\lambda)$$

where $\mathrm{End}(A^\lambda)$ is the algebra of endomorphisms of the vector space $A^\lambda$. (This theorem comes from the action of $A$ on itself!)

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The $I_\lambda$'s are called the *centrally primitive idempotents* of $A$.

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So $m_M(\lambda) = \dim(\mathrm{Hom}(A^\lambda, M))$.

Let $M$ be an $A$-module.
Define the *centralizer* of $A$ (in $\mathrm{End}(M)$) to be

$$\mathrm{End}_A(M) = \{\phi \in \mathrm{End}(M) \mid a\phi(m) = \phi(a \cdot m) \text{ for all } a \in A, m \in M\}.$$

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There is a natural action of $\mathrm{End}_A(M)$ on $\mathrm{Hom}(A^\lambda, M)$ by

$$b \cdot \phi : v \mapsto b \cdot \phi(v)$$

for all $b \in B$, $\phi \in \mathrm{Hom}(A^\lambda, M)$, and $v \in A^\lambda$.
(Check: (1) Well defined, and (2) sends $A$-mod homs to $A$-mod homs.)

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## Theorem (Double centralizer theorem)

*Let $M$ be a vector space, and $A \subseteq \mathrm{End}(M)$. Then the algebra $B = \mathrm{End}_A(M)$ is semisimple, one has $\mathrm{End}_B(M) = A$, and $M$ has the multiplicity-free complete decomposition*

$$M \cong \bigoplus_{\widehat{M}} A^\lambda \otimes B^\lambda$$

*as an $(A, B)$-bimodule, where $\{B^\lambda \mid \lambda \in \widehat{M}\}$ are distinct simple $B$-mods.*