Math 128: Lecture 19

May 9, 2014

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We're trying to calculate m_μ^λ, the dimension of $L(\lambda)_\mu$ in $L(\lambda)$, with $\lambda \in P^+ = \mathbb{Z}_{\geq 0}\{\omega_1, \ldots, \omega_r\}$.

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2. First alternative: Freudenthal's multiplicity formula.
$$m_\mu^\lambda = \frac{2}{\langle \lambda, \lambda + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle} \sum_{\alpha \in R^+} \sum_{i=1}^{\infty} \langle \mu + i\alpha, \alpha \rangle m_{\mu + i\alpha}^\lambda.$$

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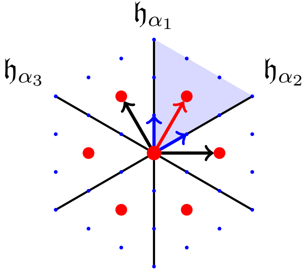
3. Second alternative: Weyl character formula. The character of a finite-dimensional $\mathfrak{g}$-module V is
$$\mathrm{ch}(V) = \sum_{\lambda \in P} \dim(V_\lambda) X^\lambda.$$
For irreducible modules, the character is given by
$$\mathrm{ch}(L(\lambda)) = \frac{a_{\lambda+\rho}}{a_\rho} \quad \text{where} \quad a_{\lambda+\rho} = \sum_{w \in W} \det(w) X^{w(\lambda+\rho)}.$$

Working example

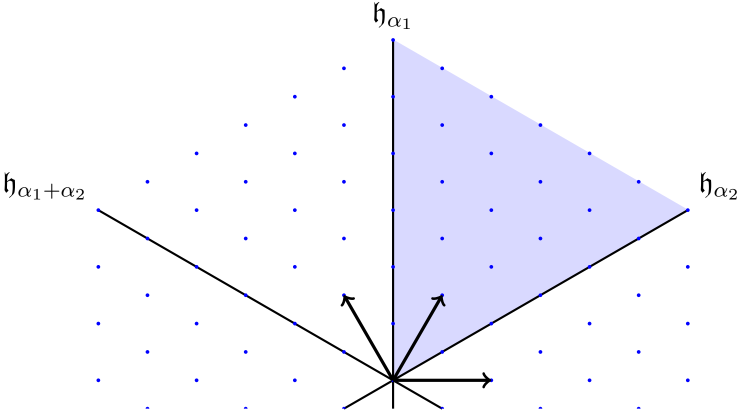
Fix $\mathfrak{g} = A_2$ with base $B = \{\beta_1, \beta_2 \mid \beta_i = \varepsilon_i - \varepsilon_{i+1}\}$.
Calculate the multiplicities of the weights in $L(\lambda)$ when

$$\lambda = \beta_1 + \beta_2 = \omega_1 + \omega_2 = \rho.$$



We have now showed $m_0^\lambda = 2$ in two ways.

Littelmann path model



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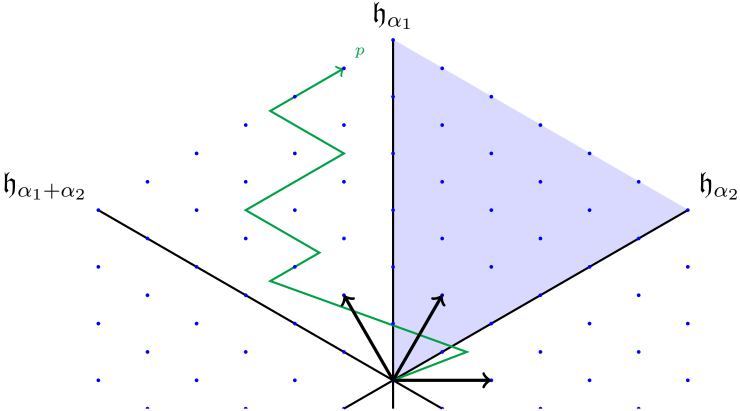

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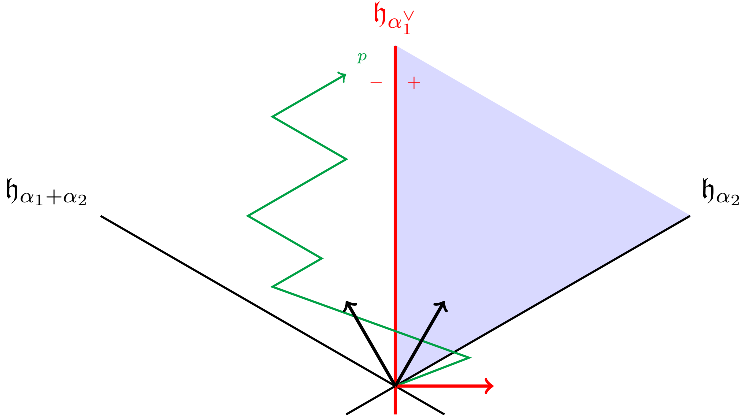

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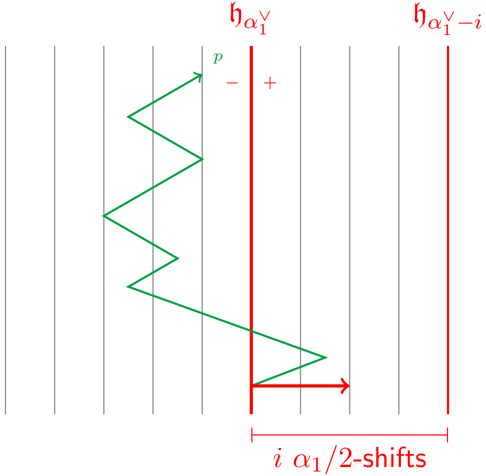

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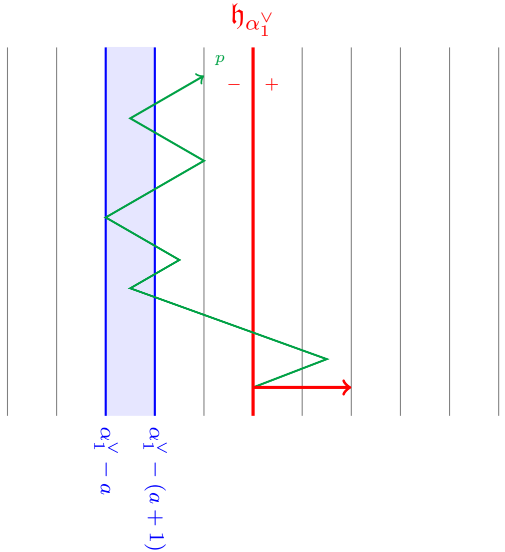

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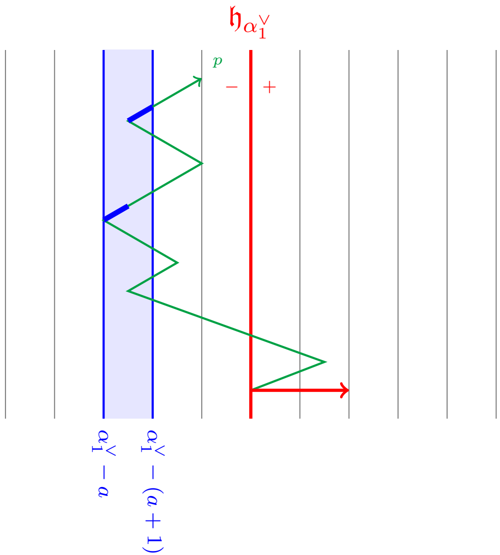

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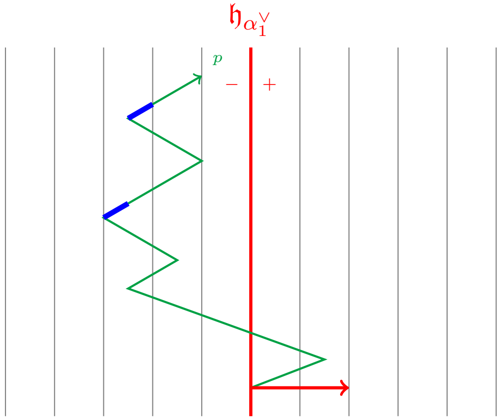

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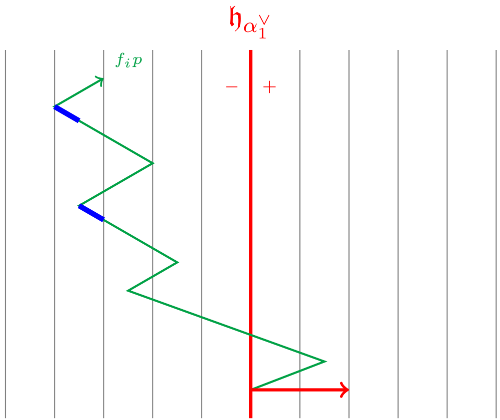

Highest weight crystals

Since $\rho = \sum_i \omega_i$, the map

$$\begin{array}{c} P^{++} \to P^{+} \\ C \cap P \to \bar{C} \cap P \end{array} \quad \text{defined by} \quad \lambda \mapsto \lambda - \rho$$

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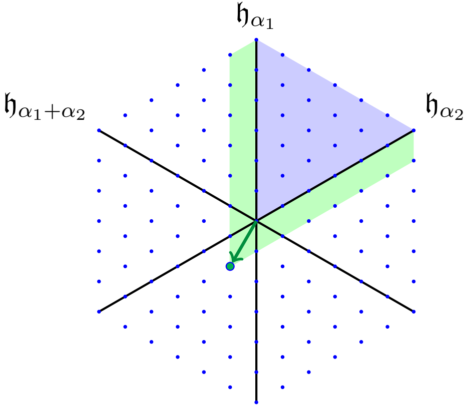

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Proposition

Let p and p' be highest weight paths of the same weight. Then the crystals generated p and p' are isomorphic.

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$$\mathrm{ch}(\mathcal{B} \sqcup \mathcal{B}') = \mathrm{ch}(\mathcal{B}) + \mathrm{ch}(\mathcal{B}').$$

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3. $\mathrm{ch}(\mathcal{B}) = \sum_{\substack{p \in \mathcal{B} \\ p \text{ is highest weight}}} \mathrm{ch}(\mathcal{B}(\mathrm{wt}(p))).$

Tensor product rules

The *concatenation* of two paths p, p' is defined by

$$pp' = \begin{cases} p(2t) & 0 \leq t \leq 1/2, \\ p(1) + p'(2(t-1/2)) & 1/2 \leq t \leq 1. \end{cases}$$

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$$\mathcal{B}(V \otimes V') = \{pp' \mid p \in \mathcal{B}(V), p' \in \mathcal{B}(V')\}.$$

2. *With $\lambda, \mu \in P^+$, and p_λ^+ highest weight in $\mathcal{B}(\lambda)$,*

$$\mathrm{ch}(L(\lambda) \otimes L(\mu)) = \sum_{\substack{q \in \mathcal{B}(\mu) \\ p_\lambda^+ q \text{ highest weight}}} \mathrm{ch}(L(\lambda + \mathrm{wt}(q))).$$