Math 128: Lecture 18

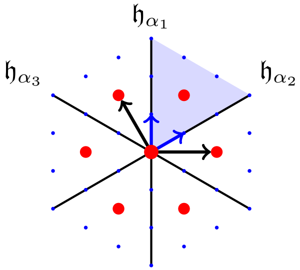
May 7, 2014

Last time:

We're trying to calculate m_μ^λ, the dimension of $L(\lambda)_\mu$ in $L(\lambda)$, with $\lambda \in P^+$.

1. Even though $\{y_1^{\ell_1} \cdots y^{\ell_m} v_\lambda^+\}$ is a spanning set of weight vectors, it's not very helpful.
2. First alternative: Freudenthal's multiplicity formula.
$$m_\mu^\lambda = \frac{2}{\langle \lambda, \lambda + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle} \sum_{\alpha \in R^+} \sum_{i=1}^{\infty} \langle \mu + i\alpha, \alpha \rangle m_{\mu + i\alpha}^\lambda.$$

Example: Fix $\mathfrak{g} = A_2$, $B = \{\beta_1, \beta_2 \mid \beta_i = \varepsilon_i - \varepsilon_{i+1}\}$, and $\lambda = \beta_1 + \beta_2 = \omega_1 + \omega_2 = \rho$.



We verified that $m_{\beta_1}^\lambda = 1$ and showed $m_0 = 2$.

Characters

Recall, with the fundamental weights $\Omega = \{\omega_i \mid i = 1, \ldots, r\}$ dual to $B^\vee$,

$$P = \mathbb{Z}\{\omega_1, \ldots, \omega_r\}, \quad P^+ = \mathbb{Z}_{\geq 0}\{\omega_1, \ldots, \omega_r\},$$

$$\text{and} \quad P^{++} = \mathbb{Z}_{>0}\{\omega_1, \ldots, \omega_r\}.$$

Characters

Recall, with the fundamental weights $\Omega = \{\omega_i \mid i = 1, \ldots, r\}$ dual to $B^\vee$,

$$P = \mathbb{Z}\{\omega_1, \ldots, \omega_r\}, \quad P^+ = \mathbb{Z}_{\geq 0}\{\omega_1, \ldots, \omega_r\},$$
$$\text{and} \quad P^{++} = \mathbb{Z}_{>0}\{\omega_1, \ldots, \omega_r\}.$$

Formally define the algebra

$$\mathbb{C}[X] = \mathbb{C}\{X^\lambda \mid \lambda \in P\} \qquad \text{with} \quad X^\lambda X^\mu = X^{\lambda+\mu}.$$

Characters

Recall, with the fundamental weights $\Omega = \{\omega_i \mid i = 1, \ldots, r\}$ dual to $B^\vee$,

$$P = \mathbb{Z}\{\omega_1, \ldots, \omega_r\}, \quad P^+ = \mathbb{Z}_{\geq 0}\{\omega_1, \ldots, \omega_r\},$$
$$\text{and} \quad P^{++} = \mathbb{Z}_{>0}\{\omega_1, \ldots, \omega_r\}.$$

Formally define the algebra

$$\mathbb{C}[X] = \mathbb{C}\{X^\lambda \mid \lambda \in P\} \qquad \text{with} \quad X^\lambda X^\mu = X^{\lambda+\mu}.$$

Let V be a finite-dimensional $\mathfrak{g}$-module. The *character* associated to V is the element of $\mathbb{C}[X]$ given by

$$\mathrm{ch}(V) = \sum_{\lambda \in P} \dim(V_\lambda) X^\lambda$$

Characters

Recall, with the fundamental weights $\Omega = \{\omega_i \mid i = 1, \dots, r\}$ dual to $B^\vee$,

$$P = \mathbb{Z}\{\omega_1, \dots, \omega_r\}, \quad P^+ = \mathbb{Z}_{\geq 0}\{\omega_1, \dots, \omega_r\},$$

$$\text{and} \quad P^{++} = \mathbb{Z}_{>0}\{\omega_1, \dots, \omega_r\}.$$

Formally define the algebra

$$\mathbb{C}[X] = \mathbb{C}\{X^\lambda \mid \lambda \in P\} \qquad \text{with} \quad X^\lambda X^\mu = X^{\lambda+\mu}.$$

Let V be a finite-dimensional $\mathfrak{g}$-module. The *character* associated to V is the element of $\mathbb{C}[X]$ given by

$$\mathrm{ch}(V) = \sum_{\lambda \in P} \dim(V_\lambda) X^\lambda$$

Example: For $\mathfrak{g} = \mathfrak{sl}_3$,

$$\mathrm{ch}(L(\beta_1 + \beta_2)) = X^{\beta_1+\beta_2} + X^{\beta_2} + X^{-\beta_1} + X^{-(\beta_1+\beta_2)} + X^{-\beta_2} + X^{\beta_1} + 2X^0.$$

Characters

W acts on P by $s_\alpha : \lambda \mapsto \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$.
This action extends to an action on $\mathbb{C}[X]$ given by

$$wX^\lambda = X^{w\lambda}.$$

Characters

W acts on P by $s_\alpha : \lambda \mapsto \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$.
This action extends to an action on $\mathbb{C}[X]$ given by

$$wX^\lambda = X^{w\lambda}.$$

Proposition

Let V, V' be finite-dimensional $\mathfrak{g}$-modules.

(1) *The character $\mathrm{ch}(V)$ is symmetric with respect to the action of W, so*

$$\mathrm{ch}(V) \in \mathbb{C}[X]^W = \{f \in \mathbb{C}[X] \mid wf = f\}.$$

(2) *One has*

$$\mathrm{ch}(V \oplus V') = \mathrm{ch}(V) + \mathrm{ch}(V') \quad \textit{and} \quad \mathrm{ch}(V \otimes V') = \mathrm{ch}(V)\mathrm{ch}(V')$$

(3) *The modules V and V' are isomorphic if and only if $\mathrm{ch}(V) = \mathrm{ch}(V')$.*

Calculating $\mathrm{ch}(L(\lambda))$

Recall $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha = \sum_{i=1}^{r} \omega_i$ and $\det(w) = (-1)^{\ell(w)}$.
Define

$$a_{\lambda+\rho} = \sum_{w \in W} \det(w) X^{w(\lambda+\rho)}.$$

Calculating $\mathrm{ch}(L(\lambda))$

Recall $\rho = \frac{1}{2}\sum_{\alpha\in R^+}\alpha = \sum_{i=1}^{r}\omega_i$ and $\det(w) = (-1)^{\ell(w)}$. Define

$$a_{\lambda+\rho} = \sum_{w\in W}\det(w)X^{w(\lambda+\rho)}.$$

Theorem (Weyl character formula)

1. *(Weyl character formula) With $\lambda \in P^+$,*

$$\mathrm{ch}(L(\lambda)) = \frac{a_{\lambda+\rho}}{a_\rho}.$$

Calculating $\mathrm{ch}(L(\lambda))$

Recall $\rho = \frac{1}{2}\sum_{\alpha\in R^+}\alpha = \sum_{i=1}^{r}\omega_i$ and $\det(w) = (-1)^{\ell(w)}$.
Define

$$a_{\lambda+\rho} = \sum_{w\in W}\det(w)X^{w(\lambda+\rho)}.$$

Theorem (Weyl character formula)

1. *(Weyl character formula) With $\lambda \in P^+$,*

$$\mathrm{ch}(L(\lambda)) = \frac{a_{\lambda+\rho}}{a_\rho}.$$

2. *(Weyl denominator formula)*

$$a_\rho = \prod_{\alpha\in R^+}(X^{\frac{1}{2}\alpha} - X^{-\frac{1}{2}\alpha}).$$

Tricks for type A_r

Recall, for $\mathfrak{g} = A_r$,

$$P^+ = \mathbb{Z}_{\geq 0}\Omega = \{\lambda_1\varepsilon_1 + \cdots + \lambda_r\varepsilon_r - \frac{|\lambda|}{r+1}\varepsilon_1 + \cdots + \varepsilon_{r+1} \mid \; **\}$$

where

$$** = \left\{\begin{matrix} \lambda_i \in \mathbb{Z}_{\geq 0}, \qquad |\lambda| = \lambda_1 + \cdots + \lambda_r \\ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0 \end{matrix}\right\}.$$

Tricks for type A_r

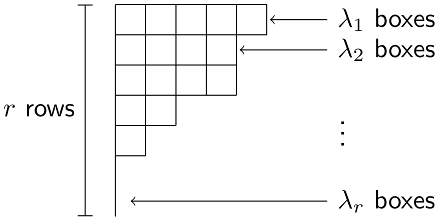
Recall, for $\mathfrak{g} = A_r$,

$$P^+ = \mathbb{Z}_{\geq 0}\Omega = \{\lambda_1\varepsilon_1 + \cdots + \lambda_r\varepsilon_r - \frac{|\lambda|}{r+1}\varepsilon_1 + \cdots + \varepsilon_{r+1} \mid \, **\}$$

where

$$** = \left\{\begin{matrix} \lambda_i \in \mathbb{Z}_{\geq 0}, & |\lambda| = \lambda_1 + \cdots + \lambda_r \\ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0 & \end{matrix}\right\}.$$

So P^+ is in bijection with integer partitions of length less than or equal to r:



Tricks for type A_r

The *weight* of a partition the collection of integers with

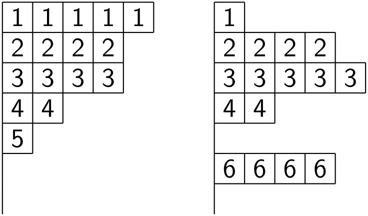
$$\lambda_1 \text{ 1's}, \quad \lambda_2 \text{ 2's}, \cdots, \lambda_r \; r\text{'s}.$$

1	1	1	1	1
2	2	2	2	
3	3	3	3	
4	4			
5				

Tricks for type A_r

The *weight* of a composition the collection of integers with

$$\lambda_1 \text{ 1's}, \quad \lambda_2 \text{ 2's}, \cdots, \lambda_r \; r\text{'s}.$$

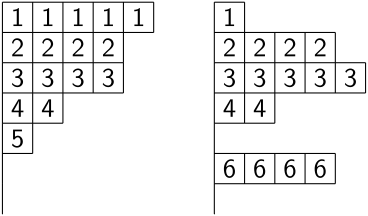


A *composition* is a partition without the condition that $\lambda_i \geq \lambda_{i+1}$.

Tricks for type A_r

The *weight* of a composition the collection of integers with

$$\lambda_1 \text{ 1's}, \quad \lambda_2 \text{ 2's}, \cdots, \lambda_r \ r\text{'s}.$$



A *composition* is a partition without the condition that $\lambda_i \geq \lambda_{i+1}$.

Let λ be a partition and μ a composition with $|\lambda| = |\mu|$.
A *semistandard tableau* or *filling* of shape λ and weight μ is a filling of the boxes in λ with the integers in $\mathrm{wt}(\mu)$ such that rows weakly increase and columns strictly increase.

Tricks for type A_r

Let λ be a partition and (abusing notation) let $L(\lambda)$ be the corresponding module.

Tricks for type A_r

Let λ be a partition and (abusing notation) let $L(\lambda)$ be the corresponding module.

The weights in P_λ can all be expressed as compositions of $|\lambda|$ of length $\leq r+1$.

Tricks for type A_r

Let λ be a partition and (abusing notation) let $L(\lambda)$ be the corresponding module.

The weights in P_λ can all be expressed as compositions of $|\lambda|$ of length $\leq r+1$.

The dimension of the weight space (corresponding to composition μ) is equal to the number semistandard fillings of λ with weight μ.

Tricks for type A_r

Let λ be a partition and (abusing notation) let $L(\lambda)$ be the corresponding module.

The weights in P_λ can all be expressed as compositions of $|\lambda|$ of length $\leq r+1$.

The dimension of the weight space (corresponding to composition μ) is equal to the number semistandard fillings of λ with weight μ.

Connection to symmetric functions land:
Let

$$x_i = X^{\varepsilon_i - \frac{1}{r+1}(\varepsilon_1 + \cdots + \varepsilon_{r+1})}, \qquad i = 1, \ldots, r+1.$$