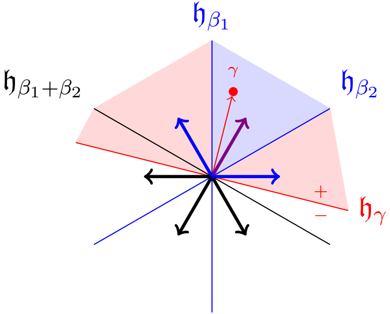
# Math 128: Lecture 16

May 2, 2014

# Last time: Existence of bases

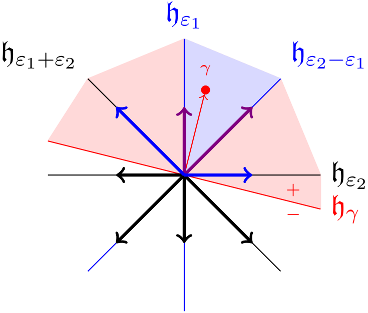
Type $A_2$ :




$$\beta_i = \varepsilon_i - \varepsilon_{i+1}$$
$$R = \{\pm\beta_1, \pm\beta_2, \pm(\beta_1+\beta_2)\}$$

$$R^+(\gamma) = \{\beta_1, \beta_2, \beta_1+\beta_2\}$$
$$B^+(\gamma) = \{\beta_1, \beta_2\}$$

Type $B_2$ :




$$\beta_1 = \varepsilon_2 - \varepsilon_1, \beta_2 = \varepsilon_1$$
$$R = \{\pm\beta_1, \pm\beta_2, \pm(\beta_1+\beta_2), \pm(\beta_1+2\beta_2)\}$$
$$R^+(\gamma) = \{\beta_1, \beta_2, \beta_1+\beta_2, \beta_1+2\beta_2\}$$
$$B^+(\gamma) = \{\beta_1, \beta_2\}$$

Fix a fundamental chamber $C$, and therefore a base $B$ and positive set of roots $R^+$. With $B = \{\beta_1, \dots, \beta_r\}$, let $s_i = s_{\beta_i}$.
Let $W$ be the group generated by $\{s_\alpha \mid \alpha \in R\}$.

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## Lemma

1. *The Weyl group $W$ is finite.* ✓
2. *The form $\langle , \rangle$ on $\mathfrak{h}_{\mathbb{R}}^*$ is $W$-invariant, i.e.* ✓
   $$\langle w(\alpha), \beta \rangle = \langle \alpha, w^{-1}(\beta) \rangle, \quad \text{for all } \alpha, \beta \in R, w \in W.$$
3. *For all $\alpha \in R$, $w \in W$, we have $w s_\alpha w^{-1} = s_{w(\alpha)}$.*
   *Also, $w(\alpha^\vee) = w(\alpha)^\vee$.* ✓
4. *The reflection associated to a simple root $\beta$ setwise fixes $R^+ - \{\beta\}$ and $R^- - \{-\beta\}$.* ✓
5. *If $w = s_{i_1} s_{i_2} \cdots s_{i_{\ell-1}}$ sends $\beta_{i_\ell}$ to a negative root, then $w s_{i_\ell} = s_{i_1} \cdots s_{i_{m-1}} s_{i_{m+1}} \cdots s_{i_{\ell-1}}$ for some $1 \leq m < \ell$.*
6. *If $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ with $\ell$ minimal, then $w(\beta_{i_\ell}) < 0$.*

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Aside: Let $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$.

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6. *If $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ with $\ell$ minimal, then $w(\beta_{i_\ell}) < 0$.*

# Big theorem on Weyl groups

## Theorem

1. *$W$ acts transitively on Weyl chambers.*
2. *Fix a base $B$. For all $\alpha \in R$ there is some $w \in W$ with $w(\alpha) \in B$.*
3. *For any base $B$, $W$ is generated by simple reflections (reflections associated to simple roots).*
4. *$W$ acts simply transitively on bases $B$ of $R$.*