Math 128: Lecture 11

April 17, 2014

Last time:

Let V be a finite-dimensional simple $\mathfrak{g}$-module. Taking $\mathfrak{sl}_2$ as a model, we will classify V as follows:

Step 1: Show that for any weight vector v, xv is also a weight vector for x a monomial in $U\mathfrak{n}^+$.

Step 2: Show the weights of xv are distinct (enough) so that there exists a $v^+ \in V$ with

$$\mathfrak{n}^+ v^+ = 0 \qquad \text{and } hv^+ = \mu(h)v^+ \text{ for some } \mu \in \mathfrak{h}^*.$$

Step 3: Show yv^+ is a weight vector for all monomials $y \in U\mathfrak{n}^-$.

Step 4: Show $xyv^+ \in U\mathfrak{n}^- v^+$ so that $V = U\mathfrak{h}^- v^+$.

Step 5: Find a good basis for V.

Step 6: Classify V in terms of μ.

Last time:

Fix a base $B = \{\beta_1, \dots, \beta_r\}$ and $R^+ = R \cap \mathbb{Z}_{\geq 0}B$.
Let V be a finite-dimensional simple $\mathfrak{g}$-module.

Last time:

Fix a base $B = \{\beta_1, \ldots, \beta_r\}$ and $R^+ = R \cap \mathbb{Z}_{\geq 0} B$.
Let V be a finite-dimensional simple $\mathfrak{g}$-module.
Step 1: For $v \in V_\lambda$, $\alpha_i \in R$, $x_i \in \mathfrak{g}_{\alpha_i}$, and $h \in \mathfrak{h}$,

$$hx_1 \cdots x_m v = \left(\lambda(h) + \sum_{i=1}^{m} \alpha_i(h) \right) x_1 \cdots x_m v. \qquad (*)$$

Last time:

Fix a base $B = \{\beta_1, \ldots, \beta_r\}$ and $R^+ = R \cap \mathbb{Z}_{\geq 0} B$.
Let V be a finite-dimensional simple $\mathfrak{g}$-module.
Step 1: For $v \in V_\lambda$, $\alpha_i \in R$, $x_i \in \mathfrak{g}_{\alpha_i}$, and $h \in \mathfrak{h}$,

$$hx_1 \cdots x_m v = \left(\lambda(h) + \sum_{i=1}^{m} \alpha_i(h)\right) x_1 \cdots x_m v. \qquad (*)$$

Define $\Omega = \{\omega_1, \ldots, \omega_r\}$ by $\langle \beta_i, \omega_j \rangle = c_j \delta_{i,j}$ for some fixed $c_j \in \mathbb{R}_{>0}$.
So for every $\alpha \in R^+$,

$$\langle \alpha, \omega_j \rangle = \sum_{i=1}^{r} z_i \langle \beta_i, \omega_j \rangle = z_j c_j \geq 0,$$

and there is some $\omega \in \Omega$ with $\langle \alpha, \omega \rangle > 0$.

Last time:

Fix a base $B = \{\beta_1, \ldots, \beta_r\}$ and $R^+ = R \cap \mathbb{Z}_{\geq 0} B$.
Let V be a finite-dimensional simple $\mathfrak{g}$-module.
Step 1: For $v \in V_\lambda$, $\alpha_i \in R$, $x_i \in \mathfrak{g}_{\alpha_i}$, and $h \in \mathfrak{h}$,

$$hx_1 \cdots x_m v = \left(\lambda(h) + \sum_{i=1}^{m} \alpha_i(h) \right) x_1 \cdots x_m v. \qquad (*)$$

Define $\Omega = \{\omega_1, \ldots, \omega_r\}$ by $\langle \beta_i, \omega_j \rangle = c_j \delta_{i,j}$ for some fixed $c_j \in \mathbb{R}_{>0}$.
So for every $\alpha \in R^+$,

$$\langle \alpha, \omega_j \rangle = \sum_{i=1}^{r} z_i \langle \beta_i, \omega_j \rangle = z_j c_j \geq 0,$$

and there is some $\omega \in \Omega$ with $\langle \alpha, \omega \rangle > 0$.
So on the basis $\{h_\omega \mid \omega \in \Omega\}$, it is clear that $\lambda + \sum_{i=1}^{m} \alpha_i$ are distinct for distinct collections $x_1, \ldots, x_m$.

Last time:

Fix a base $B = \{\beta_1, \ldots, \beta_r\}$ and $R^+ = R \cap \mathbb{Z}_{\geq 0} B$.
Let V be a finite-dimensional simple $\mathfrak{g}$-module.
Step 1: For $v \in V_\lambda$, $\alpha_i \in R$, $x_i \in \mathfrak{g}_{\alpha_i}$, and $h \in \mathfrak{h}$,

$$hx_1 \cdots x_m v = \left(\lambda(h) + \sum_{i=1}^{m} \alpha_i(h)\right) x_1 \cdots x_m v. \qquad (*)$$

Define $\Omega = \{\omega_1, \ldots, \omega_r\}$ by $\langle \beta_i, \omega_j \rangle = c_j \delta_{i,j}$ for some fixed $c_j \in \mathbb{R}_{>0}$.
So for every $\alpha \in R^+$,

$$\langle \alpha, \omega_j \rangle = \sum_{i=1}^{r} z_i \langle \beta_i, \omega_j \rangle = z_j c_j \geq 0,$$

and there is some $\omega \in \Omega$ with $\langle \alpha, \omega \rangle > 0$.
So on the basis $\{h_\omega \mid \omega \in \Omega\}$, it is clear that $\lambda + \sum_{i=1}^{m} \alpha_i$ are distinct for distinct collections $x_1, \ldots, x_m$.

Lemma (Step 2)

There is a highest weight vector $v^+ \in V$ satisfying

$$\mathfrak{n}^+ v^+ = 0 \quad \textit{and} \quad hv^+ = \mu(h)v^+ \ \textit{for some } \mu \in \mathfrak{h}^*.$$

For $v \in V_\lambda$, $\alpha_i \in R$, $x_i \in \mathfrak{g}_{\alpha_i}$, and $h \in \mathfrak{h}$,

$$hx_1 \cdots x_m v = \left(\lambda(h) + \sum_{i=1}^{m} \alpha_i(h) \right) x_1 \cdots x_m v. \qquad (*)$$

Lemma (Step 2)

There is a highest weight vector $v^+ \in V$ satisfying

$$\mathfrak{n}^+ v^+ = 0 \quad \textit{and} \quad hv^+ = \mu(h) v^+ \textit{ for some } \mu \in \mathfrak{h}^*.$$

For $v \in V_\lambda$, $\alpha_i \in R$, $x_i \in \mathfrak{g}_{\alpha_i}$, and $h \in \mathfrak{h}$,

$$hx_1 \cdots x_m v = \left(\lambda(h) + \sum_{i=1}^{m} \alpha_i(h)\right) x_1 \cdots x_m v. \qquad (*)$$

Lemma (Step 2)

There is a highest weight vector $v^+ \in V$ satisfying

$$\mathfrak{n}^+ v^+ = 0 \quad \textit{and} \quad hv^+ = \mu(h)v^+ \textit{ for some } \mu \in \mathfrak{h}^*.$$

Step 3: Show yv^+ is a weight vector for all monomials y in $U\mathfrak{n}^-$.

For $v \in V_\lambda$, $\alpha_i \in R$, $x_i \in \mathfrak{g}_{\alpha_i}$, and $h \in \mathfrak{h}$,

$$hx_1 \cdots x_m v = \left(\lambda(h) + \sum_{i=1}^{m} \alpha_i(h)\right) x_1 \cdots x_m v. \qquad (*)$$

Lemma (Step 2)

There is a highest weight vector $v^+ \in V$ satisfying

$$\mathfrak{n}^+ v^+ = 0 \quad \textit{and} \quad hv^+ = \mu(h)v^+ \textit{ for some } \mu \in \mathfrak{h}^*.$$

Step 3: Show yv^+ is a weight vector for all monomials y in $U\mathfrak{n}^-$.

Step 4: Show $xyv^+ \in U\mathfrak{n}^- v^+$ for all $x \in \mathfrak{n}^+$ and mon'ls $y \in U\mathfrak{n}^-$.

For $v \in V_\lambda$, $\alpha_i \in R$, $x_i \in \mathfrak{g}_{\alpha_i}$, and $h \in \mathfrak{h}$,

$$hx_1 \cdots x_m v = \left(\lambda(h) + \sum_{i=1}^{m} \alpha_i(h)\right) x_1 \cdots x_m v. \qquad (*)$$

Lemma (Step 2)

There is a highest weight vector $v^+ \in V$ satisfying

$$\mathfrak{n}^+ v^+ = 0 \quad \textit{and} \quad hv^+ = \mu(h)v^+ \ \textit{for some}\ \mu \in \mathfrak{h}^*.$$

Step 3: Show yv^+ is a weight vector for all monomials y in $U\mathfrak{n}^-$.

Step 4: Show $xyv^+ \in U\mathfrak{n}^- v^+$ for all $x \in \mathfrak{n}^+$ and mon'ls $y \in U\mathfrak{n}^-$.

Recall the Birkoff-Witt theorem:
Let $B = \{\beta_1, \ldots, \beta_r\}$ be a base of R with $R^+ = \{\alpha_1, \ldots, \alpha_\ell\}$. Then there are bases

$$\left\{ y_1^{m_1} \cdots y_\ell^{m_\ell} \;\middle|\; y_i \in \mathfrak{g}_{-\alpha_i}, m_i \in \mathbb{Z}_{\geq 0} \right\} \quad \text{of } U^-,$$

$$\left\{ h_{\beta_1}^{m_1} \cdots h_{\beta_r}^{m_r} \;\middle|\; m_i \in \mathbb{Z}_{\geq 0} \right\} \quad \text{of } U^0, \text{ and}$$

$$\left\{ x_1^{m_1} \cdots x_\ell^{m_\ell} \;\middle|\; x_i \in \mathfrak{g}_{\alpha_i}, m_i \in \mathbb{Z}_{\geq 0} \right\} \quad \text{of } U^+.$$

Lemma

Let V be a simple finite-dimensional $\mathfrak{g}$-module.

(a) *There is a highest weight vector $v^+ \in V$ satisfying*

$$hv^+ = \mu(h)v^+ \text{ for some } \mu \in \mathfrak{h}^*,$$
$$\mathfrak{n}^+ v^+ = 0, \quad \text{and} \quad U\mathfrak{n}^- v^+ = V.$$

(b) *V is spanned by weight vectors*

$$\{y_1^{m_1} \cdots y_\ell^{m_\ell} v^+ \mid m_i \in \mathbb{Z}_{\geq 0}\} \quad \text{with} \quad \begin{array}{l} R^+ = \{\alpha_1, \dots, \alpha_\ell\}, \text{ and} \\ y_i \in \mathfrak{g}_{-\alpha_i}, \end{array}$$

and $hyv^+ = \left(\mu - \sum_i m_i\alpha_i\right)(h)yv^+$ for $y = y_1^{m_1} \cdots y_\ell^{m_\ell}$.

(c) *The weight spaces of V are*

$$V_\lambda \qquad \text{with } \lambda = \mu - \sum_{i=1}^{r} \ell_i\beta_i, \quad \ell_i \in \mathbb{Z}_{\geq 0},$$

where $B = \{\beta_1, \dots, \beta_r\}$ is a base for the roots of $\mathfrak{g}$. In particular, $\dim(V_\mu) = 1$.

Structure of highest weight representations

When are highest weight modules simple? When are they isomorphic?

We say an element v_μ of a $\mathfrak{g}$-module M is a *primitive* element or *highest weight vector* (of weight $\mu \in \mathfrak{h}^*$) if

$$hv_\mu = \mu(h)v_\mu \qquad \text{and} \qquad \mathfrak{n}^+ v_\mu = 0.$$

We call any module generated by a primitive v_μ a *highest weight module* (of weight μ).

Structure of highest weight representations

When are highest weight modules simple? When are they isomorphic?

We say an element v_μ of a $\mathfrak{g}$-module M is a *primitive* element or *highest weight vector* (of weight $\mu \in \mathfrak{h}^*$) if

$$hv_\mu = \mu(h)v_\mu \qquad \text{and} \qquad \mathfrak{n}^+ v_\mu = 0.$$

We call any module generated by a primitive v_μ a *highest weight module* (of weight μ).

Lemma

Let M be generated by primitive v_μ.

(M is not a priori simple or finite-dimensional)

(1) *Parts (a)–(c) from the previous lemma hold.*

(2) *M is indecomposable, and therefore simple.*

(3) *There is a unique (up to scaling) primitive element of V.*

(4) *Two modules $M^{(\mu)}$ and $M^{(\lambda)}$ generated by primitive elements v_μ and v_λ, respectively, are isomorphic if and only if $\mu = \lambda$.*