

Young's Orthogonal Form for Brauer's Centralizer Algebra

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INTRODUCTION

Let G be one of the classical groups $GL(N, \mathbb{C})$, $O(N, \mathbb{C})$, $Sp(N, \mathbb{C})$ acting on the vector space $U = \mathbb{C}^N$. The question of how the n th tensor power of the representation U decomposes into irreducible summands leads to studying the centralizer $C(n, N)$ in $\text{End}(U)^{\otimes n}$ of the image of the group G . By the definition of the algebra $C(n, N)$ we have the ascending chain of subalgebras

$$C(1, N) \subset C(2, N) \subset \cdots \subset C(n, N).$$

Moreover, for the classical group G any irreducible representation of $C(n, N)$ appears at most once in the restriction of an irreducible representation of $C(n + 1, N)$. Therefore a canonical basis exists in any irreducible representation V of $C(n, N)$. Its vectors are the eigenvectors for the subalgebra $X(n, N)$ in $C(n, N)$ generated by all the central elements in the members of the above chain.

For the group $G = GL(N, \mathbb{C})$ the centralizer $C(n, N)$ is generated by the permutational action of the symmetric group $S(n)$ in $U^{\otimes n}$. The action of $S(n)$ on the vectors of the canonical basis in V was described for the first time by A. Young [Y]. G. Murphy [Mp] rederived the formulas from [Y] by using the properties of the subalgebra $X(n, N)$.

Let us now suppose that G is the orthogonal group $O(N, \mathbb{C})$. To describe the corresponding centralizer algebra $C(n, N)$ explicitly, R. Brauer [Br] introduced a certain complex associative algebra $B(n, N)$ along with a homomorphism onto $C(n, N)$. This homomorphism is injective if and only

if $N \geq n$. There is also a chain of subalgebras

$$B(1, N) \subset B(2, N) \subset \cdots \subset B(n, N).$$

The group algebra $\mathbb{C}[S(n)]$ is contained in $B(n, N)$ as a subalgebra. The structure of the algebra $B(n, N)$ was investigated by P. Hanlon and D. Wales; see [HW] and references therein. In the present article we will also work with $B(n, N)$ and regard V as a representation of the latter algebra.

For $N \geq n$ an explicit description of the action of the algebra $B(n, N)$ on the vectors of the canonical basis in V was given by J. Murakami in [Mk]. His description was based on the results of [JMO]. In the present article for any N we give a new description of this action based entirely on the properties of the commutative subalgebra $X(n, N)$ in $C(n, N)$. The case $G = Sp(N, \mathbb{C})$ is quite similar and will be considered elsewhere.

In Section 2 we introduce a remarkable family of pairwise commuting elements x_1, \dots, x_n of the algebra $B(n, N)$. For every n the element x_{n+1} belongs to the centralizer of the subalgebra $B(n, N)$ in $B(n+1, N)$. The elements x_1, \dots, x_n are the analogues of the pairwise commuting elements of $\mathbb{C}[S(n)]$ which were used in [Ju, Mp]. Their images in $C(n, N)$ belong to the subalgebra $X(n, N)$. The vectors of the canonical basis in V are eigenvectors of the elements x_1, \dots, x_n and we evaluate the respective eigenvalues; see Theorem 2.6.

There is a natural projection map $B(n+1, N) \rightarrow B(n, N)$ commuting with both left and right multiplication by the elements from $B(n, N)$; this map has been already used by H. Wenzl in [W]. The images of powers of the element x_{n+1} with respect to this map are certain central elements of the algebra $B(n, N)$. We evaluate the eigenvalues of these central elements in every irreducible representation V ; see Theorem 3.9.

The algebra $B(n, N)$ comes with family of generators $s_1, \dots, s_{n-1}; \bar{s}_1, \dots, \bar{s}_{n-1}$. The elements s_1, \dots, s_{n-1} are the standard generators of the symmetric group $S(n)$. Moreover, the quotient of the algebra $B(n, N)$ with respect to the ideal generated by $\bar{s}_1, \dots, \bar{s}_{n-1}$ is isomorphic to $\mathbb{C}[S(n)]$. We point out certain relations between the elements x_1, \dots, x_n and the generators of $B(n, N)$; see Proposition 2.3. By using Proposition 2.3 and Theorems 2.6, 3.9 we describe the action of these generators on the vectors of the canonical basis in every representation V . For the representations which factorize through $\mathbb{C}[S(n)]$ our formulas coincide with those from [Y].

In Section 4 we use the results of Sections 2 and 3 as a motivation to introduce a new algebra. This algebra is an analogue of the degenerate affine Hecke algebra $He(n)$ from [C1, C2, D]. We will denote the new algebra by $We(n, N)$ and call it the affine degenerate Wenzl algebra. The algebra $He(n)$ is a quotient of $We(n, N)$; see Corollary 4.9. For each

$m = 0, 1, 2, \dots$ the algebra $We(n, N)$ admits a homomorphism to the centralizer of the subalgebra $B(m, N)$ in $B(m + n, N)$. The kernels of all these homomorphisms have the zero intersection; see Theorem 4.7. We use these homomorphisms to construct a linear basis in the algebra $We(n, N)$; see Theorem 4.6.

1. BRAUER CENTRALIZER ALGEBRA

Let n be a positive integer and N be an arbitrary complex parameter. Denote by $\mathcal{G}(n)$ the set of all graphs with $2n$ vertices and n edges such that each vertex is incident with an edge. We will enumerate the vertices by $1, \dots, n, \bar{1}, \dots, \bar{n}$. In other words, $\mathcal{G}(n)$ consists of all partitions of the set $\{1, \dots, n, \bar{1}, \dots, \bar{n}\}$ into pairs. We will define the *Brauer algebra* $B(n, N)$ as an associative algebra over \mathbb{C} with the basic elements $b(\gamma)$, $\gamma \in \mathcal{G}(n)$.

To describe the product $b(\gamma)b(\gamma')$ in $B(n, N)$ consider the graph obtained by identifying the vertices $\bar{1}, \dots, \bar{n}$ of γ with the vertices $1, \dots, n$ of γ' , respectively. Let q be the number of loops in this graph. Remove all the loops and replace the remaining connected components by single edges, retaining the numbers of the terminal vertices. Denote by $\gamma \circ \gamma'$ the resulting graph, then by definition

$$b(\gamma)b(\gamma') = N^q \cdot b(\gamma \circ \gamma'). \tag{1.1}$$

Evidently, the dimension of $B(n, N)$ is equal to $1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n - 1)$. The algebra $B(n, N)$ contains the group algebra of the symmetric group $S(n)$; one can identify an element s of $S(n)$ with $b(\gamma)$, where the edges of γ are $\{s(1), \bar{1}\}, \dots, \{s(n), \bar{n}\}$.

An edge of the form $\{k, \bar{k}\}$ will be called *vertical*. We will regard $B(n - 1, N)$ as a subalgebra of $B(n, N)$ with the basic elements $b(\gamma)$, where γ contains the vertical edge $\{n, \bar{n}\}$. Along with a transposition (k, l) in $S(n)$ we will consider the element $\overline{(k, l)} = b(\gamma)$ of $B(n, N)$, where the only non-vertical edges of γ are $\{k, l\}$ and $\{\bar{k}, \bar{l}\}$.

We will sometimes write s_k and \bar{s}_k instead of $(k, k + 1)$ and $\overline{(k, k + 1)}$, respectively. The elements $s_1, \dots, s_{n-1}; \bar{s}_1, \dots, \bar{s}_{n-1}$ generate the algebra $B(n, N)$. One can directly verify the following relations for these elements:

$$s_k^2 = 1; \quad \bar{s}_k^2 = N\bar{s}_k; \quad s_k\bar{s}_k = \bar{s}_k s_k = \bar{s}_k; \tag{1.2}$$

$$s_k s_{k+1} s_k = s_{k+1} s_k s_{k+1}; \quad \bar{s}_k \bar{s}_{k+1} \bar{s}_k = \bar{s}_k; \quad \bar{s}_{k+1} \bar{s}_k \bar{s}_{k+1} = \bar{s}_{k+1}; \tag{1.3}$$

$$s_k \bar{s}_{k+1} \bar{s}_k = s_{k+1} \bar{s}_k; \quad \bar{s}_{k+1} \bar{s}_k s_{k+1} = \bar{s}_{k+1} s_k; \tag{1.4}$$

$$s_k s_l = s_l s_k, \quad \bar{s}_k s_l = s_l \bar{s}_k, \quad \bar{s}_k \bar{s}_l = \bar{s}_l \bar{s}_k, \quad |k - l| > 1. \tag{1.5}$$

PROPOSITION 1.1. *The relations (1.2) to (1.5) are defining relations for $B(n, N)$.*

For the proof of this proposition see [BW, Sect. 5]. Now suppose that N is a positive integer. Consider the n th tensor power of the representation $U = \mathbb{C}^N$ of the orthogonal group $G = O(N, \mathbb{C})$. Let $u(1), \dots, u(N)$ be the standard orthogonal basis in U ; let us denote by $u(i_1 \dots i_n)$ the vector $u(i_1) \otimes \dots \otimes u(i_n)$ in $U^{\otimes n}$. Consider now the centralizer algebra $C(n, N) = \text{End}_G(U^{\otimes n})$.

PROPOSITION 1.2. (a) *There is a homomorphism $B(n, N) \rightarrow C(n, N)$, where the actions of (k, l) and $(\overline{k, l})$ in $U^{\otimes n}$ for $k < l$ are defined by*

$$(k, l) \cdot u(i_1 \dots i_k \dots i_l \dots i_n) = u(i_1 \dots i_l \dots i_k \dots i_n), \tag{1.6}$$

$$(\overline{k, l}) \cdot u(i_1 \dots i_k \dots i_l \dots i_n) = \delta(i_k i_l) \cdot \sum_{i=1}^N u(i_1 \dots i \dots i \dots i_n).$$

- (b) *This homomorphism is surjective for any positive integer N .*
- (c) *This homomorphism is injective if and only if $N \geq n$.*

Proof. The actions of the elements (k, l) and $(\overline{k, l})$ in $U^{\otimes n}$ evidently commute with the action of the orthogonal group G . The parts (a) and (b) are results of [Br, Sect. 5]. The part (c) follows from [B2, Theorem 7A]. ■

The algebra $C(n, N)$ is semisimple by its definition; the irreducible representations of $C(n, N)$ are parametrized [Wy, Theorem 5.7.F] by Young diagrams with at most N boxes in the first two columns and with $n - 2r$ boxes altogether, where $r = 0, 1, \dots, [n/2]$. Denote the set of all such diagrams by $\mathcal{O}(n, N)$. Let $V(\lambda, n)$ be the representation of $C(n, N)$ corresponding to a diagram $\lambda \in \mathcal{O}(n, N)$. The next proposition is contained in [L, Theorem I]; see also [Ki, Sect. 3].

PROPOSITION 1.3. *The restriction of $V(\lambda, n)$ to $C(n - 1, N)$ decomposes into the direct sum $\bigoplus_{\mu} V(\mu, n - 1)$, where μ ranges over all the diagrams $\mu \in \mathcal{O}(n - 1, N)$ obtained from λ by removing or adding a box.*

COROLLARY 1.4. *Each irreducible representation of $C(n - 1, N)$ appears at most once in the restriction onto $C(n - 1, N)$ of an irreducible representation of $C(n, N)$.*

2. JUCYS–MURPHY ELEMENTS FOR $B(n, N)$

By definition for any complex parameter N we have the chain of subalgebras

$$B(1, N) \subset B(2, N) \subset \cdots \subset B(n, N). \quad (2.1)$$

In this section we will introduce a remarkable family of pairwise commuting elements in $B(n, N)$ corresponding to this chain; cf. [Ju, Mp]. For every $k = 1, \dots, n$ consider the element of $B(k, N)$

$$x_k = \frac{N-1}{2} + \sum_{l=1}^{k-1} (k, l) - \overline{(k, l)}. \quad (2.2)$$

LEMMA 2.1. *The element x_k commutes with any $b \in B(k-1, N)$.*

Proof. The right hand side of (2.2) is symmetric in $l = 1, \dots, k-1$. Therefore x_k commutes with any element s of $S(n-1)$. To complete the proof it suffices to check that x_k commutes with \bar{s}_{k-2} . The commutator $[\bar{s}_{k-2}, x_k]$ equals

$$\left[\overline{(k-2, k-1)}, (k-2, k) - \overline{(k-2, k)} + (k-1, k) - \overline{(k-1, k)} \right].$$

The latter commutator vanishes because

$$\begin{aligned} \overline{(k-2, k-1)} \cdot (k-2, k) &= \overline{(k-2, k-1)} \cdot \overline{(k-1, k)}, \\ \overline{(k-2, k-1)} \cdot \overline{(k-2, k)} &= \overline{(k-2, k-1)} \cdot (k-1, k), \\ (k-2, k) \cdot \overline{(k-2, k-1)} &= \overline{(k-1, k)} \cdot \overline{(k-2, k-1)}, \\ \overline{(k-2, k)} \cdot \overline{(k-2, k-1)} &= (k-1, k) \cdot \overline{(k-2, k-1)}. \end{aligned}$$

The last four equalities are verified directly by the definition (1.1). \blacksquare

COROLLARY 2.2. *The elements x_1, \dots, x_n of $B(n, N)$ pairwise commute.*

PROPOSITION 2.3. *The following relations hold in the algebra $B(n, N)$:*

$$s_k x_l = x_l s_k, \quad \bar{s}_k x_l = x_l \bar{s}_k; \quad l \neq k, k+1; \quad (2.3)$$

$$s_k x_k - x_{k+1} s_k = \bar{s}_k - 1, \quad s_k x_{k+1} - x_k s_k = 1 - \bar{s}_k; \quad (2.4)$$

$$\bar{s}_k (x_k + x_{k+1}) = 0, \quad (x_k + x_{k+1}) \bar{s}_k = 0. \quad (2.5)$$

Proof. The relations (2.3) for $l > k+1$ follow from Lemma 2.1 while those for $l < k$ follow directly from the definition (2.2). Also by this

definition we have the equality

$$x_{k+1} - s_k x_k s_k = s_k - \bar{s}_k,$$

which implies the relations (2.4). Again using (2.2) we obtain for any $l = 1, \dots, k - 1$ the equalities

$$\overline{(k, k+1)} \cdot (k, l) = \overline{(k, k+1)} \cdot \overline{(k+1, l)},$$

$$\overline{(k, k+1)} \cdot \overline{(k, l)} = \overline{(k, k+1)} \cdot (k+1, l),$$

which together with

$$\overline{(k, k+1)} \cdot (k, k+1) = \overline{(k, k+1)}, \quad \overline{(k, k+1)}^2 = N \cdot \overline{(k, k+1)}$$

imply the first relation in (2.5). The proof of the second relation is quite similar. ■

COROLLARY 2.4. *The elements $x_1^i + \dots + x_n^i$ with $i = 1, 3, \dots$ are central in $B(n, N)$.*

Proof. For any $i = 1, 2, 3, \dots$ the relations (2.4) imply that

$$s_k x_k^i = x_{k+1}^i s_k + \sum_{j=1}^i x_{k+1}^{j-1} (\bar{s}_k - 1) x_k^{i-j}, \tag{2.6}$$

$$s_k x_{k+1}^i = x_k^i s_k - \sum_{j=1}^i x_k^{j-1} (\bar{s}_k - 1) x_{k+1}^{i-j}.$$

Combining (2.5) with (2.6) gives for odd i the equalities

$$\begin{aligned} [s_k, x_k^i + x_{k+1}^i] &= \sum_{j=1}^i (x_{k+1}^{j-1} \bar{s}_k x_k^{i-j} - x_k^{j-1} \bar{s}_k x_{k+1}^{i-j}) \\ &= \sum_{j=1}^i (-1)^{j-1} (x_k^{j-1} \bar{s}_k x_k^{i-j} - x_k^{j-1} \bar{s}_k x_{k+1}^{i-j}) = 0. \end{aligned}$$

Now it follows directly from (2.3) that for odd i the sum $x_1^i + \dots + x_n^i$ commutes with s_k . This sum then also commutes with \bar{s}_k due to (2.5). ■

It follows from the definition (1.1) that for any $b \in B(k, N)$ there is a unique element $b' \in B(k-1, N)$ such that

$$\bar{s}_k b \bar{s}_k = b' \bar{s}_k; \tag{2.7}$$

cf. [W, Proposition 2.2]. Moreover, the map $b \mapsto b'$ evidently commutes with the left and right multiplication by elements from the subalgebra $B(k-1, N) \subset B(k, N)$. In particular, due to Lemma 2.1 we have

$$\bar{s}_k x_k^i \bar{s}_k = z_k^{(i)} \bar{s}_k; \quad i = 0, 1, 2, \dots, \quad (2.8)$$

where $z_k^{(0)} = N$ and $z_k^{(1)}, z_k^{(2)}, \dots$ are central elements of the algebra $B(k-1, N)$. In Section 4 we will provide explicit formulas for these elements; see the equalities (3.3) to (3.5). Here we will point out only some relations that the definition (2.8) implies.

LEMMA 2.5. *We have the relations*

$$-2z_k^{(i)} = z_k^{(i-1)} + \sum_{j=1}^i (-1)^j z_k^{(i-j)} z_k^{(j-1)}; \quad i = 1, 3, \dots \quad (2.9)$$

Proof. Let us multiply the relation (2.6) by \bar{s}_k on the left and on the right. Then due to (1.2) and (2.5) by the definition (2.8) we get

$$\begin{aligned} z_k^{(i)} \bar{s}_k &= \bar{s}_k x_{k+1}^i \bar{s}_k + \sum_{j=1}^i \bar{s}_k x_{k+1}^{j-1} (\bar{s}_k - 1) x_k^{i-j} \bar{s}_k \\ &= (-1)^i z_k^{(i)} \bar{s}_k + \sum_{j=1}^i (-1)^{j-1} (z_k^{(j-1)} z_k^{(i-j)} - z_k^{(i-1)}) \bar{s}_k. \end{aligned} \quad (2.10)$$

The last equality with odd i implies (2.9). ■

From now on until the end of Section 3 we will assume that the parameter N is a positive integer. We will then have the chain of semisimple algebras

$$C(1, N) \subset C(2, N) \subset \dots \subset C(n, N). \quad (2.11)$$

Consider the subalgebra $X(n, N)$ in $C(n, N)$ generated by all the central elements of $C(1, N), C(2, N), \dots, C(n, N)$. Each of the latter algebras is semisimple. So it follows from Corollary 1.4 that the subalgebra $X(n, N)$ is maximal commutative.

There is a canonical basis in every representation space $V(\lambda, n)$ of $C(n, N)$ corresponding to the chain (2.11); it consists of the eigenvectors of the subalgebra $X(n, N)$. The basic vectors are parametrized by the sequences

$$\Lambda = (\Lambda(1), \dots, \Lambda(n)) \in \mathcal{O}(1, N) \times \dots \times \mathcal{O}(n, N),$$

where $\Lambda(n) = \lambda$ and each two neighbouring terms of the sequence differ by exactly one box. Denote by $\mathcal{L}(\lambda, n)$ the set of all such sequences. Let $v(\Lambda)$ be the basic vector in $V(\lambda, n)$ corresponding to $\Lambda \in \mathcal{L}(\lambda, n)$. Up to a scalar multiplier, it is uniquely determined by the following condition: $v(\Lambda) \in V(\Lambda(k), k)$ in the restriction of $V(\lambda, n)$ to $C(k, N)$ for any $k = 1, \dots, n - 1$.

We will regard $V(\lambda, n)$ as a representation of the algebra $B(n, N)$ also. In the next section we will use the elements $x_1, \dots, x_n \in B(n, N)$ to describe the action of the generators $s_1, \dots, s_{n-1}; \bar{s}_1, \dots, \bar{s}_{n-1}$ of $B(n, N)$ on the vector $v(\Lambda) \in V(\lambda, n)$. It follows from Corollary 1.4 and Lemma 2.1 that the images in $C(n, N)$ of the elements x_1, \dots, x_n belong to the subalgebra $X(n, N)$. Denote by $x_k(\Lambda)$ the eigenvalue of x_k corresponding to the vector $v(\Lambda)$. For any $\Lambda \in \mathcal{L}(\lambda, n)$ we will define $\Lambda(0)$ as the empty partition.

THEOREM 2.6. *Suppose that the diagrams $\Lambda(k - 1)$ and $\Lambda(k)$ differ by the box occurring in the row i and the column j . Then*

$$x_k(\Lambda) = \pm \left(\frac{N - 1}{2} + j - i \right), \tag{2.12}$$

where the upper sign in \pm corresponds to the case $\Lambda(k) \supset \Lambda(k - 1)$ while the lower sign corresponds to $\Lambda(k) \subset \Lambda(k - 1)$.

Proof. If a box of the diagram λ occurs in the row i and the column j then the difference $j - i$ is called the *content* of this box. Denote by $n(\lambda)$ the number of the boxes in λ and by $c(\lambda)$ the sum of their contents. Due to Corollary 2.4 the element $x_1 + \dots + x_n$ is central in $B(n, N)$. We shall prove that its eigenvalue in $V(\lambda, n)$ is

$$c(\lambda, N) = \frac{N - 1}{2} n(\lambda) + c(\lambda).$$

Applying this result to k and $\Lambda(k)$, $k - 1$ and $\Lambda(k - 1)$ instead of n and λ we shall then obtain the equality (2.12).

Consider $U^{\otimes n}$ as a representation space of the algebra $G \times B(n, N)$. Due to Proposition 1.2(b, c) we then have the decomposition

$$U^{\otimes n} = \bigoplus_{\lambda \in \mathcal{O}(n, N)} U(\lambda, N) \otimes V(\lambda, n), \tag{2.13}$$

where $U(\lambda, N)$ is an irreducible representation of the group G . Here $U(\lambda, N)$ does not depend on $n \geq n(\lambda)$; see [Wy, Theorem 5.7.F]. It suffices to demonstrate that for some vector $w \in U(\lambda, N) \otimes V(\lambda, n)$

$$(x_1 + \dots + x_n) \cdot w = c(\lambda, N)w.$$

Suppose that $n = n(\lambda)$. Then due to [Wy, Theorem 5.7.F] any vector $w \in U(\lambda, N) \otimes V(\lambda, n)$ is *traceless*: we have $\overline{(k, l)} \cdot w = 0$ for all $k < l$. Thus $V(\lambda, n)$ is irreducible as a representation of the group $S(n)$; it is the representation corresponding to the diagram λ [Wy, Theorem 5.7.E]. Therefore due to [Ma, Examples I.1.3, I.7.7]

$$\begin{aligned} (x_1 + \cdots + x_n) \cdot w &= \left(\frac{N-1}{2}n + \sum_{1 \leq k < l \leq n} (k, l) \right) \cdot w \\ &= \left(\frac{N-1}{2}n + c(\lambda) \right) w; \end{aligned}$$

so that the eigenvalue of $x_1 + \cdots + x_n$ in $V(\lambda, n)$ for $n = n(\lambda)$ is equal to $c(\lambda)$.

Now suppose that $n - n(\lambda) = 2r > 0$. Then we will take $w = w_1 \otimes w_2^{\otimes r}$, where

$$\begin{aligned} w_1 &\in U(\lambda, N) \otimes V(\lambda, n(\lambda)) \subset U^{\otimes n(\lambda)}, \\ w_2 &= \sum_{i=1}^N u(i) \otimes u(i) \in U^{\otimes 2}. \end{aligned}$$

Let E_{ij} with $i, j = 1, \dots, N$ be the standard generators of the Lie algebra $\mathfrak{gl}(N)$. By definitions (1.6) and (2.2) the action in $U^{\otimes n}$ of the element $x_1 + \cdots + x_n \in B(n, N)$ coincides with that of the Casimir element

$$C = -\frac{1}{4} \sum_{i, j=1}^N (E_{ij} - E_{ji})^2$$

of the universal enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g} = \mathfrak{so}(N) \subset \mathfrak{gl}(N)$. In particular,

$$C \cdot w_2 = (x_1 + x_2) \cdot w_2 = 0.$$

But by the definition of the comultiplication of $U(\mathfrak{g})$ we have $\Delta(C) = C \otimes 1 + 1 \otimes C$. Therefore

$$\begin{aligned} (x_1 + \cdots + x_n) \cdot w &= (C \cdot w_1) \otimes w_2^{\otimes r} + w_1 \otimes (C \cdot w_2^{\otimes r}) \\ &= ((x_1 + \cdots + x_{n(\lambda)}) \cdot w_1) \otimes w_2^{\otimes r} = c(\lambda, N)w. \quad \blacksquare \end{aligned}$$

COROLLARY 2.7. *Suppose that N is odd or $N \geq 2n - 1$. Then:*

(a) *the images in $C(n, N)$ of the elements x_1, \dots, x_n generate the algebra $X(n, N)$;*

(b) *the images in $C(n, N)$ of the elements $x_1^i + \cdots + x_n^i$ with $i = 1, 3, \dots$ generate the centre of the algebra $C(n, N)$.*

Proof. Let an arbitrary diagram $\lambda \in \mathcal{O}(n, N)$ be fixed. To prove the part (a) we have to demonstrate that for all different $\Lambda \in \mathcal{L}(\lambda, n)$ the collections of eigenvalues $(x_1(\Lambda), \dots, x_n(\Lambda))$ are pairwise distinct. Suppose that $\Lambda, \Lambda' \in \mathcal{L}(n, N)$ are different; then $\Lambda(k-1) = \Lambda'(k-1)$ and $\Lambda(k) \neq \Lambda'(k)$ for some $k \in \{2, \dots, n\}$. Let $\Lambda(k), \Lambda'(k)$ differ from $\Lambda(k-1)$ by the boxes occurring in the rows i, i' and the columns j, j' , respectively. If $\Lambda(k), \Lambda'(k) \supset \Lambda(k-1)$ or $\Lambda(k), \Lambda'(k) \subset \Lambda(k-1)$ then $j-i \neq j'-i'$ and $x_k(\Lambda) \neq x_k(\Lambda')$. Suppose $\Lambda(k) \supset \Lambda(k-1)$ and $\Lambda'(k) \subset \Lambda(k-1)$; then

$$x_k(\Lambda) - x_k(\Lambda') = N - 1 + j + j' - i - i'.$$

If $j + j' \geq 3$ then $i + i' \leq N$ and $x_k(\Lambda) > x_k(\Lambda')$. If $j' = j = 1$ then $i' = i - 1$ and $x_k(\Lambda) - x_k(\Lambda') \neq 0$ for the odd N and for $N > 2i - 2$. This proves (a).

Denote by $\mathcal{E}(\lambda)$ the unordered collection of the contents of all the boxes of the diagram λ . Then λ can be uniquely restored from $\mathcal{E}(\lambda)$ since the boxes with the same content constitute the diagonals of λ and the lengths of the diagonals determine λ . To prove the part (b) we will show that the diagram $\lambda \in \mathcal{O}(n, N)$ can be uniquely restored from the collection of the eigenvalues of the elements $x_1^i + \dots + x_n^i$ with $i = 1, 3, \dots$ in $V(\lambda, n)$. Due to the equality of the formal power series in u^{-1}

$$\exp \sum_{i=1,3,\dots} 2(x_1^i + \dots + x_n^i) u^{-i}/i = \prod_{k=1}^n \frac{u + x_k}{u - x_k}$$

these eigenvalues determine the collection $\mathcal{E}(\lambda, N)$ obtained from $\mathcal{E}(\lambda)$ by removing all the pairs of the contents $j - i, j' - i'$ such that

$$\frac{N-1}{2} + j - i = - \left(\frac{N-1}{2} + j' - i' \right).$$

The latter condition implies that $j = j' = 1$ and $i + i' = N + 1$. Moreover, then $i \neq i'$ and there is only one box in each of the rows i, i' of the diagram λ . If $N \geq 2n - 1$ then $i + i' < N + 1$ so that $\mathcal{E}(\lambda) = \mathcal{E}(\lambda, N)$.

Now let N be odd. If the collection $\mathcal{E}(\lambda, N)$ does not contain $(1 - N)/2$ then there are fewer than $(N + 1)/2$ rows in λ and $i + i' \leq N - 2$ for any two different rows i, i' . Then we have $\mathcal{E}(\lambda) = \mathcal{E}(\lambda, N)$ again. Suppose that $\mathcal{E}(\lambda, N)$ does contain the number $(1 - N)/2$. Then this number is minimal in the collection $\mathcal{E}(\lambda, N)$ and occurs therein only once.

Then $\mathcal{E}(\lambda)$ is a disjoint union of $\mathcal{E}(\lambda, N)$ and

$$\left\{ 1 - i, i - N \mid \min \left(\mathcal{E}(\lambda, N) \setminus \left\{ \frac{1 - N}{2} \right\} \right) \right\} > 1 - i > \frac{1 - N}{2}.$$

Thus the collection $\mathcal{E}(\lambda)$ can be always restored from $\mathcal{E}(\lambda, N)$. This proves (b). ■

Remark. For $N = 2, 4, \dots, 2n - 2$ the statements (a) and (b) of Corollary 2.7 are no longer valid. However, the elements x_1, \dots, x_n will suffice to describe the action in $V(\lambda, n)$ of the generators $s_1, \dots, s_{n-1}; \bar{s}_1, \dots, \bar{s}_{n-1}$ of $B(n, N)$ for any positive integer N .

3. YOUNG ORTHOGONAL FORM FOR $C(n, N)$

In this section we will make explicit the matrix elements $s_k(\Lambda, \Lambda')$, $\bar{s}_k(\Lambda, \Lambda')$ of the generators $s_k, \bar{s}_k \in B(n, N)$ in the canonical basis of the representation $V(\lambda, n)$:

$$s_k \cdot v(\Lambda) = \sum_{\Lambda' \in \mathcal{L}(\lambda, n)} s_k(\Lambda, \Lambda') v(\Lambda'),$$

$$\bar{s}_k \cdot v(\Lambda) = \sum_{\Lambda' \in \mathcal{L}(\lambda, n)} \bar{s}_k(\Lambda, \Lambda') v(\Lambda').$$

Note that each of the vectors $v(\Lambda) \in V(\lambda, n)$ here is defined up to a scalar multiplier. Before specifying these multipliers we will determine the diagonal matrix elements $s_k(\Lambda, \Lambda)$ and $\bar{s}_k(\Lambda, \Lambda)$ along with the products $s_k(\Lambda, \Lambda')s_k(\Lambda', \Lambda)$ and $\bar{s}_k(\Lambda, \Lambda')\bar{s}_k(\Lambda', \Lambda)$.

Let an index $k \in \{1, \dots, n - 1\}$ and a sequence $\Lambda \in \mathcal{L}(\lambda, n)$ be fixed. Denote by $V(\Lambda, k)$ the subspace in $V(\lambda, n)$ spanned by the vectors $v(\Lambda')$ such that $\Lambda'(l) = \Lambda(l)$ for any $l \neq k$. The action of s_k and \bar{s}_k in $V(\lambda, n)$ preserves this subspace.

PROPOSITION 3.1. *Suppose $\Lambda(k - 1) \neq \Lambda(k + 1)$. Then $\bar{s}_k \cdot v(\Lambda) = 0$.*

Proof. The diagrams $\Lambda(k - 1)$ and $\Lambda(k + 1)$ differ by two boxes; let $j - i$ and $j' - i'$ be the contents of these boxes. If $x_k(\Lambda) + x_{k+1}(\Lambda) \neq 0$ then by applying to $v(\Lambda)$ the first of the relations (2.5) we obtain that $\bar{s}_k \cdot v(\Lambda) = 0$.

Now suppose that $x_k(\Lambda) + x_{k+1}(\Lambda) = 0$. Then by Theorem 2.6 we have $j = j' = 1$ and $i + i' = N + 1$. Therefore either $\Lambda(k - 1) \subset \Lambda(k + 1)$ or $\Lambda(k - 1) \supset \Lambda(k + 1)$. In both of these cases the action of the elements s_k and \bar{s}_k in $V(\lambda, n)$ preserves the subspace $\mathbb{C} \cdot v(\Lambda)$. Moreover, then we have $x_k(\Lambda) = \frac{1}{2}$ and $x_{k+1}(\Lambda) = -\frac{1}{2}$. Due to (1.2) by applying the first of the relations (2.4) to the vector $\bar{s}_k \cdot v(\Lambda)$ we obtain that $(N - 2)\bar{s}_k \cdot v(\Lambda) = 0$. This equality completes the proof for $N \neq 2$.

Let ϕ and ψ denote respectively the empty diagram and the diagram consisting of the two boxes in the first column. If $N = 2$ then

$$\{\Lambda(k - 1), \Lambda(k + 1)\} = \{\phi, \psi\}.$$

The representation $U(\psi, 2)$ of the group $G = O(2, \mathbb{C})$ is the determinant representation and

$$U(\Lambda(k + 1), 2) = U(\Lambda(k - 1), 2) \otimes U(\psi, 2).$$

Therefore the action of \bar{s}_k in the space

$$\begin{aligned} & \text{Hom}_{B(k-1, 2)}(V(\Lambda(k - 1), k - 1), V(\Lambda(k + 1), k + 1)) \\ &= \text{Hom}_{G \times B(k-1, 2)}(U(\Lambda(k + 1), 2) \otimes V(\Lambda(k - 1), k - 1), U^{\otimes(k+1)}) \\ &= \text{Hom}_G(U(\Lambda(k + 1), 2), U(\Lambda(k - 1), 2) \otimes U^{\otimes 2}) \end{aligned}$$

coincides with that of \bar{s}_1 in $V(\psi, 2)$. This proves that $\bar{s}_k \cdot v(\Lambda) = 0$ for $N = 2$. ■

PROPOSITION 3.2. *Suppose that $\Lambda(k - 1) \neq \Lambda(k + 1)$. Then we have $x_k(\Lambda) \neq x_{k+1}(\Lambda)$ and $s_k(\Lambda, \Lambda) = (x_{k+1}(\Lambda) - x_k(\Lambda))^{-1}$.*

Proof. By applying to the vector $v(\Lambda)$ the second of the relations (2.4) we obtain that $s_k(\Lambda, \Lambda)(x_{k+1}(\Lambda) - x_k(\Lambda)) = 1$. ■

Observe that if $\Lambda(k - 1) \neq \Lambda(k + 1)$ then the space $V(\Lambda, k)$ has dimension at most two. Therefore due to the relation $s_k^2 = 1$ we get

COROLLARY 3.3. *Suppose $\Lambda(k - 1) \neq \Lambda(k + 1)$ and $v(\Lambda') \in V(\Lambda, k)$ with $\Lambda \neq \Lambda'$. Then $s_k(\Lambda, \Lambda')s_k(\Lambda', \Lambda) = 1 - (x_{k+1}(\Lambda) - x_k(\Lambda))^{-2}$.*

Two Young diagrams are *associated* if the sum of the lengths of their first columns equals N while the lengths of their other columns respectively coincide. In particular, for even N a diagram is *self-associated* if its first column consists of $N/2$ boxes.

LEMMA 3.4. *For any $v(\Lambda') \in V(\Lambda, k)$ we have $x_k(\Lambda) + x_k(\Lambda') \neq 0$ unless N is odd and $\Lambda' = \Lambda$, where the diagrams $\Lambda(k - 1), \Lambda(k)$ are associated.*

Proof. Let the diagrams $\Lambda(k)$ and $\Lambda'(k)$ differ from $\Lambda(k-1)$ by the boxes with contents $j-i$ and $j'-i'$, respectively. If

$$\Lambda(k) \subset \Lambda(k-1) \subset \Lambda'(k) \text{ or } \Lambda(k) \supset \Lambda(k-1) \supset \Lambda'(k)$$

then $j-i \neq j'-i'$ and $x_k(\Lambda) + x_k(\Lambda') \neq 0$.

Assume now that $\Lambda(k), \Lambda'(k) \subset \Lambda(k-1)$ or $\Lambda(k), \Lambda'(k) \supset \Lambda(k-1)$. Then the condition $x_k(\Lambda) + x_k(\Lambda') = 0$ takes the form

$$j-i+j'-i' = 1-N, \quad (3.1)$$

which implies that $j=j'=1$. Then $i=i'$ due to our assumption. Thus we obtain from (3.1) that $\Lambda = \Lambda'$ and $N = 2i+1$ so that the diagrams $\Lambda(k-1), \Lambda(k)$ are associated. ■

Let us now consider the case $\Lambda(k-1) = \Lambda(k+1)$. Due to Theorem 2.6 we then have $x_k(\Lambda') + x_{k+1}(\Lambda') = 0$ for any $v(\Lambda') \in V(\Lambda, k)$. The next two lemmas are contained in [RW, Theorem 2.4(b)]. We will include their short proofs here.

LEMMA 3.5. *Suppose that $\Lambda(k-1) = \Lambda(k+1)$. Then*

$$\bar{s}_k(\Lambda, \Lambda) = \frac{\dim U(\Lambda(k), N)}{\dim U(\Lambda(k+1), N)}.$$

Proof. Let τ_n denote the trace on the algebra $C(n, N)$ induced by the usual matrix trace on $\text{End}(U^{\otimes n})$. We will also regard τ_n as a trace on the algebra $B(n, N)$. Then due to the definition (1.1)

$$\tau_{k+1}(\bar{s}_k b \bar{s}_k) = N \cdot \tau_k(b), \quad b \in B(k, N).$$

Let us apply this equality to an element $b \in B(k, N)$ such that for any $\Lambda' \in \mathcal{L}(\lambda, n)$

$$b \cdot v(\Lambda') = \begin{cases} v(\Lambda') & \text{if } \Lambda'(l) = \Lambda(l) \text{ for } l \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

Due to Proposition 3.1 we then obtain that

$$N \cdot \dim U(\Lambda(k+1), N) \bar{s}_k(\Lambda, \Lambda) = N \cdot \dim U(\Lambda(k), N). \quad \blacksquare$$

LEMMA 3.6. *Suppose that $\Lambda(k-1) = \Lambda(k+1)$. Then the image of the action of \bar{s}_k in the subspace $V(\Lambda, k)$ is one-dimensional.*

Proof. Due to the relation $\bar{s}_k^2 = N\bar{s}_k$ any eigenvalue of the action of \bar{s}_k in the subspace $V(\Lambda, k)$ is either N or zero. Lemma 3.5 along with the decomposition

$$U(\Lambda(k+1), N) \otimes U = \bigoplus_{v(\Lambda') \in V(\Lambda, k)} U(\Lambda'(k), N)$$

implies that the trace of this action is equal to N . Therefore the eigenvalue N of the action of \bar{s}_k in $V(\Lambda, k)$ has multiplicity one. ■

COROLLARY 3.7. *Suppose that $\Lambda(k - 1) = \Lambda(k + 1)$ and the vector $v(\Lambda') \in V(\Lambda, k)$. Then $\bar{s}_k(\Lambda, \Lambda')\bar{s}_k(\Lambda', \Lambda) = \bar{s}_k(\Lambda, \Lambda)\bar{s}_k(\Lambda', \Lambda')$.*

There are well known explicit formulas for the dimension of the irreducible representation $U(\lambda, N)$ of the orthogonal group G ; see, for instance, [EK, Sect. 3]. Due to Lemma 3.5 these formulas already provide certain expressions for the matrix element $\bar{s}_k(\Lambda, \Lambda)$. In this section we will employ the relations (2.4) and (2.8) to determine $\bar{s}_k(\Lambda, \Lambda)$ independently of any explicit formula for $\dim U(\lambda, N)$.

Suppose that $\Lambda(k - 1) = \Lambda(k + 1) = \mu$. Let l be the number of pairwise distinct rows (or columns) in the diagram μ . Then one can obtain $l + 1$ diagrams by adding a box to μ and l diagrams by removing a box from μ . Let c_1, \dots, c_{l+1} and d_1, \dots, d_l be the contents of these boxes, respectively. Denote by b_1, \dots, b_{2l+1} the numbers

$$\begin{aligned} & (N - 1)/2 + c_1, \dots, (N - 1)/2 + c_{l+1}, \\ & -(N - 1)/2 - d_1, \dots, -(N - 1)/2 - d_l \end{aligned}$$

taken in an arbitrary order. Introduce the formal power series in u^{-1}

$$Q(\mu, u) = \sum_{i \geq 0} q_i(\mu)u^{-i} = \prod_{j=1}^{2l+1} \frac{u + b_j}{u - b_j}; \tag{3.2}$$

the coefficients $q_1(\mu), q_2(\mu), \dots$ are the symmetric *Schur q -functions* in b_1, \dots, b_{2l+1} .

Further, let m be the total number of boxes in the diagram μ . Let e_1, \dots, e_m be the contents of all these boxes. Denote by a_1, \dots, a_m the numbers

$$(N - 1)/2 + e_1, \dots, (N - 1)/2 + e_m$$

taken in an any order. Then we have another expression for the series (3.2).

LEMMA 3.8. *We have the equality*

$$Q(\mu, u) = \frac{u + (N - 1)/2}{u - (N - 1)/2} \cdot \prod_{j=1}^m \frac{(u + a_j)^2 - 1}{(u - a_j)^2 - 1} \frac{(u - a_j)^2}{(u + a_j)^2}.$$

Proof. For any $h \in \mathbb{C}$ and $i \geq 0$ we have the equality

$$\begin{aligned} & \sum_{j=1}^{l+1} (h + c_j)^i - \sum_{j=1}^l (h + d_j)^i \\ &= h^i + \sum_{j=1}^m (h + e_j + 1)^i - 2(h + e_j)^i + (h + e_j - 1)^i, \end{aligned}$$

which can be verified by induction on m . Let us multiply each side of this equality by $2u^{-i}/i$ and take the sum over all odd i . Exponentiate the resulting sums. When $h = (N - 1)/2$ we get the required statement. ■

Denote by $z_k^{(i)}(\mu)$ the eigenvalue in the representation $V(\mu, k - 1)$ of the central element $z_k^{(i)} \in B(k - 1, N)$ defined by (2.8). Introduce the generating function

$$Z(\mu, u) = \sum_{i \geq 0} z_k^{(i)}(\mu) u^{-i}.$$

THEOREM 3.9. *We have the equality*

$$Z(\mu, u) = (u + \frac{1}{2}) \cdot Q(\mu, u) - u + \frac{1}{2}.$$

Proof. Consider the generating series

$$Z_k(u) = \sum_{i \geq 0} z_k^{(i)} u^{-i} \in B(k - 1, N)[[u^{-1}]]. \quad (3.3)$$

Then determine the series $Q_k(u) \in B(k - 1, N)[[u^{-1}]]$ by the equality

$$Z_k(u) = (u + \frac{1}{2}) \cdot Q_k(u) - u + \frac{1}{2}. \quad (3.4)$$

We have to prove that the eigenvalue of $Q_k(u)$ in $V(\mu, k - 1)$ is exactly $Q(\mu, u)$. Later on in a more general setting we shall prove the equality

$$Q_k(u) = \frac{u + (N - 1)/2}{u - (N - 1)/2} \cdot \prod_{l=1}^{k-1} \frac{(u + x_l)^2 - 1}{(u - x_l)^2 - 1} \frac{(u - x_l)^2}{(u + x_l)^2}; \quad (3.5)$$

see Corollary 4.3. Due to (3.5) the required statement follows from Theorem 2.6 and Lemma 3.8. ■

COROLLARY 3.10. *Suppose that $\Lambda(k - 1) = \Lambda(k + 1) = \mu$ and let $x_k(\Lambda) = b$. Then*

$$\bar{s}_k(\Lambda, \Lambda) = \begin{cases} (2b + 1) \prod_{b_j \neq b} \frac{b + b_j}{b - b_j} & \text{if } b \neq -1/2; \\ - \prod_{b_j \neq b} \frac{b + b_j}{b - b_j} & \text{if } b = -1/2. \end{cases}$$

Proof. Let us make use of the relation

$$\bar{s}_k \cdot Z_k(u)u^{-1} = \bar{s}_k(u - x_k)^{-1}\bar{s}_k. \tag{3.6}$$

Any eigenvalue of x_k in $V(\Lambda, k)$ distinct from $-\frac{1}{2}$ has multiplicity one. Moreover, it then appears only once in the collection b_1, \dots, b_{2l+1} . Therefore if $b \neq -\frac{1}{2}$ we get from (3.6)

$$\bar{s}_k(\Lambda, \Lambda) = \operatorname{res}_{u=b} Z(\mu, u)/u = (2b + 1) \prod_{b_j \neq b} \frac{b + b_j}{b - b_j};$$

here we use Theorem 3.9 and the inequality $\bar{s}_k(\Lambda, \Lambda) \neq 0$ provided by Lemma 3.5.

If $b = -\frac{1}{2}$ appears as an eigenvalue of x_k in $V(\Lambda, k)$ it has multiplicity two. Then it appears twice in the collection b_1, \dots, b_{2l+1} . Let $x_k(\Lambda) = x_k(\Lambda') = -\frac{1}{2}$, where $v(\Lambda') \in V(\Lambda, k)$ and $\Lambda(k) \neq \Lambda'(k)$. Then the diagrams $\Lambda(k)$ and $\Lambda'(k)$ are associated. Representations $U(\Lambda(k), N)$ and $U(\Lambda'(k), N)$ of G have the same dimension [Wy, Theorem 5.9.A]. So $\bar{s}_k(\Lambda, \Lambda) = \bar{s}_k(\Lambda', \Lambda') \neq 0$ by Lemma 3.5. Therefore

$$\bar{s}_k(\Lambda, \Lambda) = \operatorname{res}_{u=b} Z(\mu, u)/2u = - \prod_{b_j \neq b} \frac{b + b_j}{b - b_j}. \quad \blacksquare$$

Let $\delta(\Lambda, \Lambda')$ be the Kronecker delta.

PROPOSITION 3.11. *Suppose that $\Lambda(k - 1) = \Lambda(k + 1)$ and the vector $v(\Lambda') \in V(\Lambda, k)$. Then*

$$s_k(\Lambda, \Lambda') = (\bar{s}_k(\Lambda, \Lambda') - \delta(\Lambda, \Lambda'))(x_k(\Lambda) + x_k(\Lambda'))^{-1} \tag{3.7}$$

unless N is odd and $\Lambda' = \Lambda$, where the diagrams $\Lambda(k), \Lambda(k - 1)$ are associated. In the latter case $s_k(\Lambda, \Lambda) = 1$.

Proof. Due to the equality $x_k(\Lambda') + x_{k+1}(\Lambda') = 0$ the first of the relations (2.4) implies

$$s_k(\Lambda, \Lambda')(x_k(\Lambda) + x_k(\Lambda')) = (\bar{s}_k(\Lambda, \Lambda') - \delta(\Lambda, \Lambda')).$$

Thus when $x_k(\Lambda) + x_k(\Lambda') \neq 0$ we obtain the equality (3.7).

Now assume that $x_k(\Lambda) + x_k(\Lambda') = 0$. Then by Lemma 3.4 the number N is odd and $\Lambda' = \Lambda$, where the diagrams $\Lambda(k), \Lambda(k - 1)$ are associated. Then $\bar{s}_k(\Lambda, \Lambda) = 1$ by Lemma 3.5 and $x_k(\Lambda) = 0$. Moreover, all the eigenvalues of the element x_k in $V(\Lambda, k)$ are then pairwise distinct. Consider the diagonal matrix element of the relation $s_k \bar{s}_k = \bar{s}_k$ in $V(\Lambda, k)$ corresponding to the vector $v(\Lambda)$. By making use of the equality (3.7) and

of Corollary 3.7 we obtain that

$$s_k(\Lambda, \Lambda) + \sum_{\Lambda'' \neq \Lambda} \bar{s}_k(\Lambda'', \Lambda'')/x_k(\Lambda'') = 1, \tag{3.8}$$

where $v(\Lambda'') \in V(\Lambda, k)$. If $x_k(\Lambda'') = b \neq 0$ then

$$\bar{s}_k(\Lambda'', \Lambda'')/x_k(\Lambda'') = \operatorname{res}_{u=b} Z(\mu, u)/u^2$$

while by Theorem 3.9 the residues of $Z(\mu, u)/u^2$ at $u = 0, \infty$ equal zero. Therefore

$$\sum_{\Lambda'' \neq \Lambda} \bar{s}_k(\Lambda'', \Lambda'')/x_k(\Lambda'') = 0.$$

By comparing this equality with (3.8) we complete the proof of Proposition 3.11. ■

Now let the index k run through the set $\{1, \dots, n - 1\}$ while the sequences Λ, Λ' run through the set $\mathcal{L}(\lambda, n)$. If $v(\Lambda') \notin V(\Lambda, k)$ then $s_k(\Lambda, \Lambda') = \bar{s}_k(\Lambda, \Lambda') = 0$.

Suppose that $v(\Lambda') \in V(\Lambda, k)$. As we have already mentioned, the vectors $v(\Lambda), v(\Lambda') \in V(\lambda, n)$ are defined up to scalar multipliers. Up to the choice of these multipliers Proposition 3.1 and Corollaries 3.7, 3.10 describe the matrix element $\bar{s}_k(\Lambda, \Lambda')$ while Propositions 3.2, 3.10 and Corollary 3.3 describe the matrix element $s_k(\Lambda, \Lambda')$. The following theorem completes the description of these matrix elements.

THEOREM 3.12. *Suppose that $v(\Lambda') \in V(\Lambda, k)$ and $\Lambda \neq \Lambda'$. Then one can assume*

$$s_k(\Lambda, \Lambda') = s_k(\Lambda', \Lambda) > 0 \quad \text{if } \Lambda(k - 1) \neq \Lambda(k + 1), \tag{3.9}$$

$$\bar{s}_k(\Lambda, \Lambda') = \bar{s}_k(\Lambda', \Lambda) > 0 \quad \text{if } \Lambda(k - 1) = \Lambda(k + 1). \tag{3.10}$$

Proof. Let us demonstrate first that for all k and Λ, Λ' we can assume the equalities

$$s_k(\Lambda, \Lambda') = s_k(\Lambda', \Lambda) \quad \text{and} \quad \bar{s}_k(\Lambda, \Lambda') = \bar{s}_k(\Lambda', \Lambda). \tag{3.11}$$

Consider the non-degenerate symmetric bilinear form F on $U^{\otimes n}$ which is the product of the standard forms on the factors U . The action of the group G in $U^{\otimes n}$ preserves F and the direct summands in (2.13) are orthogonal with respect to F . The restriction of F onto the direct summand $U(\lambda, N) \otimes V(\lambda, n)$ splits into the product of a G -invariant bilinear symmetric form on $U(\lambda, N)$ and of a certain form on $V(\lambda, N)$. The vectors

$v(\Lambda) \in V(\lambda, n)$ are orthogonal with respect to the latter form. The actions of s_k and \bar{s}_k in $U^{\otimes n}$ are self-adjoint with respect to F . Therefore the vectors $v(\Lambda)$ can be so chosen that the equalities (3.11) hold.

We will now assume that the equalities in (3.9) and (3.10) do hold. The matrix elements in these equalities then belong to $\mathbb{R} \setminus \{0\}$ by Corollaries 3.3 and 3.7. We have to prove that the vectors $v(\Lambda)$ can be so chosen that the inequalities in (3.9) and (3.10) also hold. Let an index $k \in \{1, \dots, n - 2\}$ and a sequence $\Lambda \in \mathcal{L}(\lambda, n)$ be fixed. Denote by $V(\Lambda, k, k + 1)$ the subspace in $V(\lambda, n)$ spanned by the vectors $v(\Lambda')$ such that $\Lambda'(l) = \Lambda(l)$ for any $l \neq k, k + 1$. The action of s_k, s_{k+1} and \bar{s}_k, \bar{s}_{k+1} in $V(\lambda, n)$ preserves this subspace.

Note that due to Proposition 1.1 the generators s_k, \bar{s}_k of the algebra $B(n, N)$ are *local* in the sense [V1, V2]: the only relations between s_k, \bar{s}_k and s_l, \bar{s}_l with $|k - l| > 1$ are the commutation relations (1.5). Therefore it suffices to only choose the vectors $v(\Lambda') \in V(\Lambda, k, k + 1)$ so that for every two distinct $v(\Lambda'), v(\Lambda'') \in V(\Lambda, k, k + 1)$

$$s_k(\Lambda', \Lambda'') > 0 \quad \text{if } \Lambda'(k + 1) = \Lambda''(k + 1) \neq \Lambda(k - 1), \quad (3.12)$$

$$s_{k+1}(\Lambda', \Lambda'') > 0 \quad \text{if } \Lambda'(k) = \Lambda''(k) \neq \Lambda(k + 2); \quad (3.13)$$

$$\bar{s}_k(\Lambda', \Lambda'') > 0 \quad \text{if } \Lambda'(k + 1) = \Lambda''(k + 1) = \Lambda(k - 1), \quad (3.14)$$

$$\bar{s}_{k+1}(\Lambda', \Lambda'') > 0 \quad \text{if } \Lambda'(k) = \Lambda''(k) = \Lambda(k + 2). \quad (3.15)$$

The diagrams $\Lambda(k - 1)$ and $\Lambda(k + 2)$ may differ either by one or by three boxes. Let us consider the latter case; cf. [Mo]. In this case we have $\Lambda'(k + 1) \neq \Lambda(k - 1)$ and $\Lambda'(k) \neq \Lambda(k + 2)$ for any vector $v(\Lambda')$ from $V(\Lambda, k, k + 1)$. Furthermore, then $\dim V(\Lambda, k, k + 1) \in \{1, 3, 6\}$. We will treat each of these possibilities separately.

(i) If $\dim V(\Lambda, k, k + 1) = 1$ then we can choose the vector $v(\Lambda)$ arbitrarily.

(ii) Suppose that $\dim V(\Lambda, k, k + 1) = 3$. Initially let us make an arbitrary choice of the basic vectors $v(\Lambda), v(\Lambda'), v(\Lambda'') \in V(\Lambda, k, k + 1)$. We can assume that

$$\Lambda(k) \neq \Lambda'(k) = \Lambda''(k) \quad \text{and} \quad \Lambda'(k + 1) \neq \Lambda(k + 1) = \Lambda''(k + 1).$$

Then it suffices to take instead of $v(\Lambda')$ and $v(\Lambda'')$ respectively the vectors

$$v(\Lambda') \cdot \text{sign } s_k(\Lambda, \Lambda'') s_{k+1}(\Lambda', \Lambda'') \quad \text{and} \quad v(\Lambda'') \cdot \text{sign } s_k(\Lambda, \Lambda').$$

(iii) Suppose that $\dim V(\Lambda, k, k + 1) = 6$. Initially let us again make an arbitrary choice of the basic vectors in $V(\Lambda, k, k + 1)$,

$$v(\Lambda), v(\Omega), v(\Omega'), v(\Lambda'), v(\Lambda''), v(\Omega''). \quad (3.16)$$

We can assume that

$$\begin{aligned}\Lambda(k) &= \Omega'(k), & \Lambda'(k) &= \Omega(k), \\ \Lambda(k+1) &= \Omega(k+1), & \Lambda'(k+1) &= \Omega''(k+1), \\ & & \Lambda''(k) &= \Omega''(k), \\ & & \Lambda''(k+1) &= \Omega'(k+1).\end{aligned}$$

By applying the first relation in (1.3) to the vector $v(\Lambda)$ and taking the coefficient at $v(\Omega'')$ we get the equality

$$s_k(\Lambda, \Omega) s_{k+1}(\Lambda', \Omega) s_k(\Lambda', \Omega'') = s_{k+1}(\Lambda, \Omega') s_k(\Lambda'', \Omega') s_{k+1}(\Lambda'', \Omega'').$$

Now it suffices to take instead of the vectors (3.16) respectively the vectors

$$\begin{aligned}v(\Lambda), & \quad v(\Omega) \cdot \text{sign } s_k(\Lambda, \Omega), & v(\Omega') \cdot \text{sign } s_{k+1}(\Lambda, \Omega'), \\ & v(\Lambda') \cdot \text{sign } s_k(\Lambda, \Omega) s_{k+1}(\Lambda', \Omega), \\ & v(\Lambda'') \cdot \text{sign } s_{k+1}(\Lambda, \Omega') s_k(\Lambda'', \Omega'), \\ & v(\Omega'') \cdot \text{sign } s_k(\Lambda, \Omega) s_{k+1}(\Lambda', \Omega) s_k(\Lambda', \Omega'').\end{aligned}$$

Finally, let us consider the case when the diagrams $\Lambda(k-1)$ and $\Lambda(k+2)$ differ by only one box. Since $V(\Lambda, k, k+1) = V(\Lambda', k, k+1)$ for any $\Lambda' \in \mathcal{L}(\lambda, n)$ such that $\Lambda(l) = \Lambda'(l)$ with $l \neq k, k+1$ we can assume that

$$\Lambda(k) = \Lambda(k+2) \quad \text{and} \quad \Lambda(k+1) = \Lambda(k-1). \quad (3.17)$$

Let us make any choice of every basic vector $v(\Lambda') \in V(\Lambda, k, k+1)$. Consider any vector $v(\Omega) \in V(\Lambda, k, k+1)$ such that

$$\Omega(k) \neq \Lambda(k) \quad \text{and} \quad \Omega(k+1) \neq \Lambda(k+1).$$

Then consider the vectors $v(\Omega'), v(\Omega'') \in V(\Lambda, k, k+1)$ such that

$$\begin{aligned}\Omega'(k) &= \Lambda(k), & \Omega''(k) &= \Omega(k), & \Omega'(k+1) &= \Omega(k+1), \\ & & \Omega''(k+1) &= \Lambda(k+1).\end{aligned}$$

By applying the first relation in (1.4) to the vector $v(\Lambda)$ and taking the coefficient at $v(\Omega)$ we get the equality

$$\bar{s}_k(\Lambda, \Lambda) \bar{s}_{k+1}(\Lambda, \Omega') s_k(\Omega, \Omega') = \bar{s}_k(\Lambda, \Omega'') s_{k+1}(\Omega, \Omega'').$$

Since $\bar{s}_k(\Lambda, \Lambda) > 0$ by Lemma 3.5, this equality implies that

$$\text{sign } \bar{s}_{k+1}(\Lambda, \Omega') s_k(\Omega, \Omega') = \text{sign } \bar{s}_k(\Lambda, \Omega'') s_{k+1}(\Omega, \Omega'').$$

We will keep to the initial choice of the vector $v(\Lambda)$ and replace each $v(\Omega)$ by

$$v(\Omega) \cdot \text{sign } \bar{s}_{k+1}(\Lambda, \Omega') s_k(\Omega, \Omega'). \tag{3.18}$$

For $\Lambda' \neq \Lambda$, where $\Lambda'(k) = \Lambda(k)$ or $\Lambda'(k + 1) = \Lambda(k + 1)$, we will replace $v(\Lambda')$ by

$$\begin{aligned} v(\Lambda') \cdot \text{sign } \bar{s}_k(\Lambda, \Lambda') & \quad \text{if } \Lambda'(k + 1) = \Lambda(k + 1), \\ v(\Lambda') \cdot \text{sign } \bar{s}_{k+1}(\Lambda, \Lambda') & \quad \text{if } \Lambda'(k) = \Lambda(k). \end{aligned}$$

Due to Corollary 3.7 the latter replacement will make all the matrix elements of \bar{s}_k in $V(\Lambda, k)$ and \bar{s}_{k+1} in $V(\Lambda, k + 1)$ positive. But for any $v(\Lambda') \in V(\Lambda, k, k + 1)$

$$\begin{aligned} \Lambda'(k + 1) = \Lambda(k - 1) & \Rightarrow v(\Lambda') \in V(\Lambda, k), \\ \Lambda'(k) = \Lambda(k + 2) & \Rightarrow v(\Lambda') \in V(\Lambda, k + 1) \end{aligned}$$

due to (3.17). So the inequalities (3.14), (3.15) in $V(\Lambda, k, k + 1)$ will then be satisfied. Further, for any two distinct $v(\Lambda'), v(\Lambda'') \in V(\Lambda, k, k + 1)$

$$\begin{aligned} \Lambda'(k + 1) = \Lambda''(k + 1) \neq \Lambda(k - 1) & \Rightarrow \Lambda'(k) = \Lambda(k) \text{ or} \\ & \Lambda''(k) = \Lambda(k), \\ \Lambda'(k) = \Lambda''(k) \neq \Lambda(k + 2) & \Rightarrow \Lambda'(k + 1) = \Lambda(k + 1) \text{ or} \\ & \Lambda''(k + 1) = \Lambda(k + 1). \end{aligned}$$

So by the replacement of each $v(\Omega)$ by the respective vector (3.18) the inequalities (3.12), (3.13) in $V(\Lambda, k, k + 1)$ will be satisfied. Theorem 3.12 is proved. ■

4. DEGENERATE AFFINE WENZL ALGEBRA

In this section again we assume that N is an arbitrary complex number. We will now use the results of Section 2 as a motivation to introduce a new object. This is the complex associative algebra generated by the algebra $B(n, N)$ along with the pairwise commuting elements y_1, \dots, y_n and central elements w_1, w_2, \dots subjected to the following relations. We impose the relations

$$s_k y_l = y_l s_k, \quad \bar{s}_k y_l = y_l \bar{s}_k; \quad l \neq k, k + 1; \tag{4.1}$$

$$s_k y_k - y_{k+1} s_k = \bar{s}_k - 1, \quad s_k y_{k+1} - y_k s_k = 1 - \bar{s}_k; \tag{4.2}$$

$$\bar{s}_k (y_k + y_{k+1}) = 0, \quad (y_k + y_{k+1}) \bar{s}_k = 0. \tag{4.3}$$

Moreover, we impose the relations

$$\bar{s}_1 y_1^i \bar{s}_1 = w_i \bar{s}_1; \quad i = 1, 2, \dots \quad (4.4)$$

We will view this algebra as an analogue of the degenerate affine Hecke algebra $He(n)$ considered in [C1, C2, D]; see Corollary 4.9 below. We will denote the above introduced algebra by $We(n, N)$ and call it the *degenerate affine Wenzl algebra* in honour of H. Wenzl, who has used the maps

$$B(k, N) \rightarrow B(k-1, N): \quad b \mapsto b'; \quad k = 1, 2, \dots, n-1$$

defined by (2.7) to prove that the algebra $B(n, N)$ is semisimple when N is not an integer.

It is convenient to put $w_0 = N$. The equality (4.4) is then valid for $i = 0$ also. The assignments

$$y_k \mapsto x_k, \quad w_i \mapsto z_1^{(i)}$$

define a homomorphism

$$\pi: We(n, N) \rightarrow B(n, N) \quad (4.5)$$

identical on $B(n, N)$ by the relations (2.3) to (2.5) and (2.8). Moreover, in the proofs of Corollary 2.4 and Lemma 2.5 we used not the definition (2.2) of the elements x_1, \dots, x_n but the latter relations. Therefore the relations (4.1) to (4.4) imply that

$$-2w_i = w_{i-1} + \sum_{j=1}^i (-1)^j w_{i-j} w_{j-1}; \quad i = 1, 3, \dots \quad (4.6)$$

In particular, we have $w_1 = N(N-1)/2$. Moreover, the following proposition holds.

PROPOSITION 4.1. *The elements $y_1^i + \dots + y_n^i$ with $i = 1, 3, \dots$ are central in the algebra $We(n, N)$.*

We have the ascending chain of algebras $We(1, N) \subset We(2, N) \subset \dots$ by definition.

PROPOSITION 4.2. *For each $k = 1, 2, \dots$ we have the equalities*

$$\bar{s}_k y_k^i \bar{s}_k = w_k^{(i)} \bar{s}_k; \quad i = 0, 1, 2, \dots, \quad (4.7)$$

where $w_k^{(i)}$ is a central element of the algebra $We(k-1, N)$. The generating series

$$W_k(u) = \sum_{i \geq 0} w_k^{(i)} u^{-i}$$

satisfy

$$\frac{W_{k+1}(u) + u - \frac{1}{2}}{W_k(u) + u - \frac{1}{2}} = \frac{(u + y_k)^2 - 1}{(u - y_k)^2 - 1} \cdot \frac{(u - y_k)^2}{(u + y_k)^2}. \quad (4.8)$$

Proof. We use the induction on k . The equalities (4.7) hold for $k = 1$ and $w_1^{(i)} = w_i$ by definition. Assume that the equalities (4.7) are valid for $k = 1, \dots, n$ and that the corresponding series $W_1(u), \dots, W_n(u)$ satisfy (4.8). Due to Proposition 4.1 it then suffices to verify the equalities (4.7) and (4.8) for $k = n + 1$ and $k = n$, respectively. We will work with the formal power series in u^{-1}

$$\sum_{i \geq 0} y_k^i u^{-i} = \frac{u}{u - y_k}; \quad k = n, n + 1.$$

We will also use the following corollary to the first relation in (4.2) and the second relation in (4.3) with $k = n$:

$$s_n \frac{1}{u - y_n} = \frac{1}{u - y_{n+1}} s_n + \frac{1}{u + y_n} \bar{s}_n \frac{1}{u - y_n} - \frac{1}{(u - y_n)(u - y_{n+1})}. \quad (4.9)$$

By multiplying the equality (4.9) on the left by \bar{s}_n and replacing u by $-u$ we get

$$\bar{s}_n \frac{1}{u - y_n} s_n = \bar{s}_n \frac{1}{u + y_n} + \bar{s}_n \frac{W_n(u)}{u(u + y_n)} - \bar{s}_n \frac{1}{u^2 - y_n^2} \quad (4.10)$$

by the inductive assumption. Let us now multiply the equality (4.9) on the right by s_n and use (4.9) once more along with (4.10). We then obtain

$$\begin{aligned} s_n \frac{1}{u - y_n} s_n &= \frac{(u - y_n)^2 - 1}{(u - y_n)^2 (u - y_{n+1})} - \frac{1}{u - y_n} s_n \frac{1}{u - y_n} \\ &+ \frac{1}{u + y_n} \bar{s}_n \frac{1}{u + y_n} + \frac{1}{u^2 - y_n^2} \bar{s}_n \frac{1}{u - y_n} \\ &+ \frac{1}{u + y_n} \bar{s}_n \frac{W_n(u)}{u(u + y_n)} - \frac{1}{u + y_n} \bar{s}_n \frac{1}{u^2 - y_n^2}. \end{aligned}$$

By multiplying the right-hand side of this equality by \bar{s}_{n+1} on the left and right we get

$$\begin{aligned} & \frac{(u - y_n)^2 - 1}{(u - y_n)^2} \cdot \bar{s}_{n+1} \frac{1}{u - y_{n+1}} \bar{s}_{n+1} \\ & + \frac{W_n(u)}{u(u + y_n)^2} \bar{s}_{n+1} + \frac{2(1 - 2u)y_n}{(u^2 - y_n^2)^2} \bar{s}_{n+1} \end{aligned} \quad (4.11)$$

due to the relations $\bar{s}_{n+1}y_n = y_n\bar{s}_{n+1}$ and $\bar{s}_{n+1}s_n\bar{s}_{n+1} = \bar{s}_{n+1}\bar{s}_n\bar{s}_{n+1} = \bar{s}_{n+1}$. On the other hand, since $s_{n+1}y_n s_{n+1} = y_n$ we have by the inductive assumption

$$\bar{s}_{n+1}s_n \frac{1}{u - y_n} s_n \bar{s}_{n+1} = \frac{W_n(u)}{u} \bar{s}_{n+1}.$$

By comparing the last expression with (4.11) we get

$$\begin{aligned} & \frac{(u - y_n)^2 - 1}{(u - y_n)^2} \cdot \bar{s}_{n+1} \frac{u}{u - y_{n+1}} \bar{s}_{n+1} \\ & = \frac{(u + y_n)^2 - 1}{(u + y_n)^2} W_n(u) \bar{s}_{n+1} - \frac{2u(1 - 2u)y_n}{(u^2 - y_n^2)^2} \bar{s}_{n+1}. \end{aligned}$$

The latter equality implies that

$$\bar{s}_{n+1} \frac{u}{u - y_{n+1}} \bar{s}_{n+1} = W_{n+1}(u) \bar{s}_{n+1},$$

where the series $W_{n+1}(u)$ satisfies (4.8) for $k = n$. Proposition 4.2 is now proved. ■

Consider the series $Z_k(u)$ and $Q_k(u)$ defined by (3.3) and (3.4), respectively. Since $x_1 = (N - 1)/2$ we have

$$Q_1(u) = \frac{u + (N - 1)/2}{u - (N - 1)/2}.$$

Furthermore,

$$\pi: W_k(u) \mapsto Z_k(u)$$

for $k = 1, 2, \dots$ by (4.7). Thus we obtain a corollary to Proposition 4.2.

COROLLARY 4.3. *For every $k = 1, 2, \dots$ the equality (3.5) holds.*

In the remaining part of this section will construct a linear basis in the algebra $We(n, N)$. Let us equip the algebra $We(n, N)$ with an ascending filtration by defining the degrees of its generators in the following way:

$$\deg s_k = \deg \bar{s}_k = 0, \quad \deg y_k = 1, \quad \deg w^{(i)} = 0.$$

Denote by u_k the image of the element $y_k \in We(n, N)$ in the corresponding graded algebra $gr We(n, N)$. In the latter algebra by the relations (4.1) to (4.3) we have

$$su_k s^{-1} = u_{s(k)}, \quad s \in S(n). \tag{4.12}$$

These relations along with (4.1) and (4.3) imply that

$$\overline{(k, l)}u_m = u_m\overline{(k, l)}, \quad m \neq k, l; \tag{4.13}$$

$$\overline{(k, l)} \cdot (u_k + u_l) = 0, \quad (u_k + u_l) \cdot \overline{(k, l)} = 0; \quad k \neq l. \tag{4.14}$$

Furthermore, due to the relations (4.4) and (4.12) we have

$$\overline{(k, l)}u_k^i\overline{(k, l)} = 0; \quad i = 1, 2, \dots; k \neq l. \tag{4.15}$$

By definition, the elements $b(\gamma)$, where γ runs through the set of graphs $\mathcal{G}(n)$, constitute a linear basis in the algebra $B(n, N)$. Any edge of a graph $\gamma \in \mathcal{G}(n)$ of the form $\{k, l\}$ or $\{\bar{k}, \bar{l}\}$ will be called *horizontal*. If $k < l$ then the vertex k or \bar{k} will be called the *left end* of the horizontal edge $\{k, l\}$ or $\{\bar{k}, \bar{l}\}$, respectively. The vertex l or \bar{l} will be then called the *right end*.

The number of horizontal edges in a graph $\gamma \in \mathcal{G}(n)$ is even. If this number is $2r$, the element $b(\gamma) \in B(n, N)$ has the form $\overline{(k_1, l_1)} \cdots \overline{(k_r, l_r)} \cdot s$, where $s \in S(n)$ and all $k_1, l_1, \dots, k_r, l_r$ are pairwise distinct. The elements $b(\gamma)$, where the graph γ has $2r$ horizontal edges or more, span a two-sided ideal in $B(n, N)$.

LEMMA 4.4. *Let u be a monomial in u_1, \dots, u_n . For any two graphs $\gamma, \gamma' \in \mathcal{G}(n)$ we have the equality in the algebra $gr We(n, N)$*

$$b(\gamma)ub(\gamma') = \varepsilon \cdot u'b(\gamma)b(\gamma')u'', \tag{4.16}$$

where $\varepsilon \in \{1, 0, -1\}$ and u', u'' are certain monomials in u_1, \dots, u_n .

Proof. Let $2r$ and $2r'$ be the numbers of horizontal edges in the graphs γ and γ' , respectively. We will employ the induction on the minimum of r, r' and on the degree of u . If each of these two numbers is zero we have nothing to prove.

Suppose that $r, r' \geq 1$ and $u \neq 1$. By the relations (4.12) we can assume that

$$b(\gamma) = \overline{(k_1, l_1)} \cdots \overline{(k_r, l_r)} = b,$$

$$b(\gamma') = \overline{(k'_1, l'_1)} \cdots \overline{(k'_r, l'_r)} = b',$$

where $k_1, l_1, \dots, k_r, l_r$ are pairwise distinct and so are $k'_1, l'_1, \dots, k'_r, l'_r$. Consider the monomial $u = u_1^{i_1} \cdots u_n^{i_n}$. Choose any index $k \in \{1, \dots, n\}$ such that $i_k \neq 0$. If

$$k \notin \{k_1, l_1, \dots, k_r, l_r\} \quad \text{or} \quad k \notin \{k'_1, l'_1, \dots, k'_r, l'_r\}$$

then respectively

$$bu_k^{i_k} = u_k^{i_k}b \quad \text{or} \quad u_k^{i_k}b' = b'u_k^{i_k}$$

by (4.13). Then we obtain the equality (4.16) by the inductive assumption.

Now suppose that $k = k_j = k'_{j'}$ for some j and j' . Denote $l_j = l$ and $l_{j'} = l'$. Let b'' denote the product obtained from b' by removing the factor $\overline{(k, l')}$. If $l = l'$ then by the relations (4.13) to (4.15) we have

$$bub' = (-1)^{i_l} bu_k^{i_k+i_l} \cdot \prod_{m \neq k, l} u_m^{i_m} \cdot \overline{(k, l)} b''$$

$$= (-1)^{i_l} bu_k^{i_k+i_l} \overline{(k, l)} \cdot \prod_{m \neq k, l} u_m^{i_m} \cdot b'' = 0.$$

Suppose that $l \neq l'$. Then by the relations (4.13) and (4.14) we have

$$bub' = (-1)^{i_k} bu_l^{i_k+i_{l'}} \cdot \prod_{m \neq k, l'} u_m^{i_m} \cdot \overline{(k, l')} b''$$

$$= (-1)^{i_k} b \overline{(k, l')} u_l^{i_k+i_{l'}} \cdot \prod_{m \neq k, l'} u_m^{i_m} \cdot b''.$$

We have $b(\gamma'') = b \overline{(k, l')}$ for a certain graph $\gamma'' \in \mathcal{G}(n)$. The number of horizontal edges in the latter graph equals $2r$. Thus we obtain the equality (4.16) again by the inductive assumption. ■

Consider any graph $\gamma \in \mathcal{G}(n)$ with exactly $2r$ horizontal edges. Let

$$k_1, \dots, k_r, \bar{k}'_1, \dots, \bar{k}'_r \quad \text{and} \quad l_1, \dots, l_r, \bar{l}'_1, \dots, \bar{l}'_r$$

be all the left ends and the right ends of the horizontal edges, respectively.

LEMMA 4.5. For any two monomials u and u' in u_1, \dots, u_n we have the equality in the algebra $\text{gr We}(n, N)$

$$ub(\gamma)u' = \varepsilon \cdot u_1^{i_1} \cdots u_n^{i_n} b(\gamma) u_1^{j_1} \cdots u_n^{j_n},$$

where $\varepsilon \in \{1, 0, -1\}$ and

$$k \in \{l_1, \dots, l_r\} \Rightarrow i_k = 0; \quad j_k \neq 0 \Rightarrow k \in \{l'_1, \dots, l'_r\}. \quad (4.17)$$

Proof. By (4.12) we can assume that $b(\gamma) = \overline{(k_1, l_1)} \cdots \overline{(k_r, l_r)}$,

$$k_1 = k'_1, \dots, k_r = k'_r \quad \text{and} \quad l_1 = l'_1, \dots, l_r = l'_r.$$

The required statement follows then directly from the relations (4.13) and (4.14). ■

Any product in the algebra $We(n, N)$ of the form

$$y_1^{i_1} \cdots y_n^{i_n} b(\gamma) y_1^{j_1} \cdots y_n^{j_n} \cdot w_2^{h_2} w_4^{h_4} \cdots \quad (4.18)$$

will be called a *regular monomial* if the exponents i_1, \dots, i_n and j_1, \dots, j_n satisfy the conditions (4.17). The two theorems below are the main results of this section.

THEOREM 4.6. *All the regular monomials (4.18) constitute a basis in $We(n, N)$.*

By the relations (2.3) to (2.5) and (2.8) for every $m = 0, 1, 2, \dots$ the assignments

$$s_k \mapsto s_{m+k}, \quad \bar{s}_k \mapsto \bar{s}_{m+k}, \quad y_k \mapsto x_{m+k}, \quad w_i \mapsto z_{m+1}^{(i)}$$

define a homomorphism

$$\pi_m: We(n, N) \rightarrow B(m + n, N).$$

In particular, the homomorphism π_0 coincides with (4.5). Furthermore, by Lemma 2.1 the image of the homomorphism π_m commutes with the subalgebra $B(m, N)$ in $B(m + n, N)$.

THEOREM 4.7. *The kernels of $\pi_0, \pi_1, \pi_2, \dots$ have zero intersection.*

Due to Lemmas 4.4, 4.5 and to the equalities (4.6) every element of the algebra $We(n, N)$ can be expressed as a linear combination of regular monomials. Thus Theorems 4.6 and 4.7 both follow from the next lemma; cf. [O, Lemma 2.1.11]. Fix any finite set \mathcal{F} of regular monomials (4.18). Let m be the maximum of the sums

$$i_1 + j_1 + \cdots + i_n + j_n + 2h_2 + 4h_4 + \cdots$$

corresponding to the monomials from the set \mathcal{F} . Consider any linear

combination $F \in We(n, N)$ of some monomials from \mathcal{F} with non-zero coefficients.

LEMMA 4.8. *Suppose that $\pi_m(F) = 0$. Then monomials on which the maximum m is attained do not appear in F .*

Proof. Due to Corollary 4.3 $Q_{m+1}(u)$ equals

$$\exp \sum_{i=1,3,\dots} 2 \left((N-1)^i / 2^i + \sum_{j=1}^m (x_j + 1)^i - 2x_j^i + (x_j - 1)^i \right) u^{-i}/i.$$

Therefore for each $i = 2, 4, \dots$ by (3.4) the element $z_{m+1}^{(i)} \in B(m, N)$ is a symmetric polynomial in x_1, \dots, x_m of the form

$$2i(x_1^{i-1} + \dots + x_m^{i-1}) + \text{terms of smaller degrees.}$$

Consider the subset $\mathcal{F}' \subset \mathcal{F}$ formed by all the monomials where the maximum m is attained. Then consider the subset $\mathcal{F}'' \subset \mathcal{F}'$ formed by the monomials with the minimal number of horizontal edges in the corresponding graphs γ . Let $2r$ be that minimum. It suffices to prove that the monomials from \mathcal{F}'' do not appear in F .

Choose any regular monomial (4.18) from the set \mathcal{F} . The image of this monomial with respect to the homomorphism π_m is a certain linear combination f of elements $b(\Gamma) \in B(m+n, N)$, where $\Gamma \in \mathcal{G}(m+n)$. Denote by \mathcal{S} the subset in $\mathcal{G}(m+n)$ consisting of those graphs which have:

- exactly $2r$ horizontal edges;
- no vertical edges of the form $\{k, \bar{k}\}$, where $k \leq m$;
- no horizontal edges of the form $\{k, l\}$ or $\{\bar{k}, \bar{l}\}$, where $k, l \leq m$.

Consider the terms of f corresponding to the graphs $\Gamma \in \mathcal{S}$. Such terms appear in f only if the chosen monomial belongs to the subset \mathcal{F}'' . Suppose this is the case. Then among those terms are the products

$$\begin{aligned} & \prod_{k=1}^n (m_{k1}, m+k) \cdots (m_{ki_k}, m+k) \cdot \pi_m(b(\gamma)) \\ & \times \prod_{k=1}^n (m'_{k1}, m+k) \cdots (m'_{kj_k}, m+k) \\ & \cdot \prod_{i=2,4,\dots} \prod_{j=1}^{h_i} 2(m_{ij1}, \dots, m_{iji}), \end{aligned} \tag{4.19}$$

where the juxtaposition of the sequences

$$m_{k1}, \dots, m_{ki_k}, m'_{k1}, \dots, m'_{kj_k}; \quad k = 1, \dots, n$$

and

$$m_{ij_1}, \dots, m_{ij_i}; \quad j = 1, \dots, h_i; \quad i = 2, 4, \dots$$

runs through the set of all permutations of the sequence $1, 2, \dots, m$. All these terms will be called the *leading terms* of f . Note that the parameters

$$\gamma, i_1, \dots, i_n, j_1, \dots, j_n, h_2, h_4, \dots$$

can be uniquely restored from any of these leading terms.

All the non-leading terms of f corresponding to graphs $\Gamma \in \mathcal{G}$ can be obtained from the products (4.19) by certain non-empty sets of the following replacements. One can replace the factor in (4.19)

$$(m_{k1}, m + k) \cdots (m_{ki_k}, m + k) \text{ by } \overline{(m_{k1}, m + k)} \cdots \overline{(m_{ki_k}, m + k)} \cdot (-1)^{i_k}$$

provided the vertex k of the graph γ is the left end of a horizontal edge. One can also replace any factor

$$(m'_{k1}, m + k) \cdots (m'_{kj_k}, m + k) \text{ by } \overline{(m'_{k1}, m + k)} \cdots \overline{(m'_{kj_k}, m + k)} \cdot (-1)^{j_k}.$$

Due to the conditions (4.17) the terms so obtained are not proportional to any leading term in the image with respect to π_m of any monomial from \mathcal{F}^n . This observation completes the proof. ■

We will now compare the algebra $We(n, N)$ with the *degenerate affine Hecke algebra* $He(n)$ from [C1, C2, D]. The latter algebra is generated by the group algebra $\mathbb{C}[S(n)]$ and the pairwise commuting elements v_1, \dots, v_n subjected to the relations

$$s_k v_l = v_l s_k, \quad l \neq k, k + 1; \\ s_k v_k - v_{k+1} s_k = -1, \quad s_k v_{k+1} - v_k s_k = 1.$$

By (4.1) to (4.4) we have the following corollary to Theorem 4.6.

COROLLARY 4.9. *For any $f_2, f_4, \dots \in \mathbb{C}$ the assignments $s_k \mapsto s_k, \bar{s}_k \mapsto 0, y_k \mapsto v_k$, and*

$$w_i \mapsto f_i, \quad i = 2, 4, \dots$$

determine a homomorphism of the algebra $We(n, N)$ onto $He(n)$.

The subalgebra in $He(n)$ generated by the elements v_1, \dots, v_n is maximal commutative. The centre of the algebra $He(n)$ consists of all symmetric polynomials in v_1, \dots, v_n . For the proofs of these two statements see [C2, Sect. 1]. The next corollary provides analogues of these statements for the algebra $We(n, N)$.

COROLLARY 4.10. *The subalgebra in $We(n, N)$ generated by the elements y_1, \dots, y_n and w_1, w_2, \dots is maximal commutative. The elements $y_1^i + \dots + y_n^i$ with $i = 1, 3, \dots$ and w_i with $i = 2, 4, \dots$ generate the centre of the algebra $We(n, N)$.*

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