Introduction to Quantum Groups and Crystal Bases

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Introduction

The notion of a *quantum group* was introduced by V. G. Drinfel′d and M. Jimbo, independently, in their study of the quantum Yang-Baxter equation arising from two-dimensional solvable lattice models (**[10, 23]**). Quantum groups are certain families of Hopf algebras that are deformations of universal enveloping algebras of Kac-Moody algebras. Over the past 20 years, they turned out to be the fundamental algebraic structure behind many branches of mathematics and mathematical physics such as:

(1) solvable lattice models in statistical mechanics,

(2) topological invariant theory of links and knots,

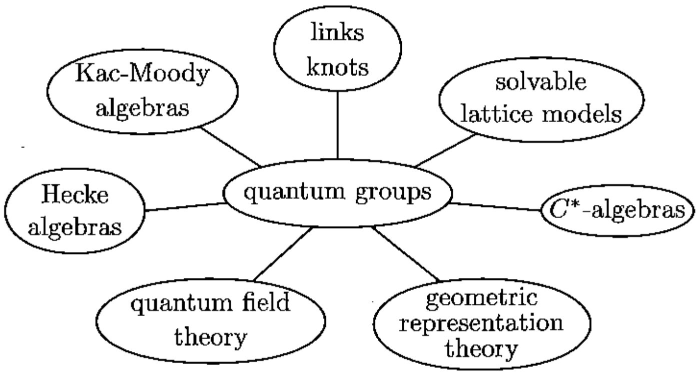
(3) representation theory of Kac-Moody algebras,

(4) representation theory of algebraic structures, e.g., Hecke algebra,

(5) topological quantum field theory,

(6) geometric representation theory,

(7) C^*-algebras.



In particular, the theory of *crystal bases* or *canonical bases* developed independently by M. Kashiwara and G. Lusztig provides a powerful combinatorial and geometric tool to study the representations of quantum groups ([**38, 39, 48**]). The purpose of this book is to provide an elementary introduction to the theory of quantum groups and crystal bases focusing on the combinatorial aspects of the theory.

In such an introductory book, the first question to be answered would be: *What are quantum groups?* In his famous lecture given at the International Congress of Mathematicians held at Berkeley in 1986, Drinfel′d gave a *definition* of quantum groups: it was defined to be the *spectrum of a certain Hopf algebra* [**11**]. That is, Drinfel′d noted that any suitable category of groups (algebraic, topological, etc.) is antiequivalent to a suitable category of *commutative* Hopf algebras. In such a situation, one goes from the group to the algebra by considering a suitable algebra of functions, while the group can be reconstructed by taking the *spectrum* in the sense of Grothendieck. Thus, even when one has a noncommutative Hopf algebra, it becomes natural to think of the corresponding object in the opposite category as a *quantum group*, and this is the meaning of Drinfel′d's definition.

In this book, we focus on the quantum groups that appear as certain deformations of universal enveloping algebras of Kac-Moody algebras. For example, let $\mathfrak{g}$ be a finite dimensional simple Lie algebra, and let $U(\mathfrak{g})$ be its universal enveloping algebra. Choose a generic parameter q. Then, for each q, we can associate a Hopf algebra $U_q(\mathfrak{g})$, called the *quantum group* or the *quantized universal enveloping algebra*, whose structure *tends to* that of $U(\mathfrak{g})$ as q approaches 1. Therefore, we get a family of Hopf algebras $U_q(\mathfrak{g})$, and when $q = 1$, it is the same as the Hopf algebra $U(\mathfrak{g})$.

The following example shows how one can understand the above statement in a naive way. This example is not rigorous, not even mathematical, but it gives us a certain intuition. Let $\mathfrak{g} = \mathfrak{sl}_2$ be the complex Lie algebra of 2×2 matrices of trace 0. It is generated by the elements e, f, and h with defining relations

$$[e,f] = h, \qquad [h,e] = 2e, \qquad [h,f] = -2f.$$

Thus its universal enveloping algebra $U(\mathfrak{sl}_2)$ is an associative algebra over $\mathbf{C}$ with 1 generated by the elements e, f, and h with defining relations

$$ef - fe = h, \qquad he - eh = 2e, \qquad hf - fh = -2f.$$

Now, the quantum group $U_q(\mathfrak{g}) = U_q(\mathfrak{sl}_2)$ is defined to be the associative algebra over $\mathbf{C}(q)$ with 1 generated by the elements e, f, and q^h with defining relations

$$ef - fe = \frac{q^h - q^{-h}}{q - q^{-1}}, \qquad q^h e q^{-h} = q^2 e, \qquad q^h f q^{-h} = q^{-2} f.$$

Let us look at the first of these defining relations. As q approaches 1, the left-hand side remains the same as $ef - fe$, but the right-hand side is undetermined. If we apply L'Hospital's rule (however absurd it might be), then the right-hand side is equal to

$$\lim_{q\to 1} \frac{q^h - q^{-h}}{q - q^{-1}} = \lim_{q\to 1} \frac{hq^{h-1} + hq^{-h-1}}{1 + q^{-2}} = \frac{2h}{2} = h,$$

as desired.

For the second relation, if we let $q \to 1$, then we get $e = e$, which gives nothing new. But if we *differentiate* both sides with respect to q (again, however absurd it might be), we get

$$hq^{h-1}eq^{-h} + q^h e(-h)q^{-h-1} = 2qe.$$

Thus, if we take the limit $q \to 1$, we get

$$he - eh = 2e.$$

Similarly, the last relation gives the desired relation as $q \to 1$.

Therefore, one can say that for each generic parameter q, there is a quantum group $U_q(\mathfrak{sl}_2)$ which is a Hopf algebra, so we have a family of Hopf algebras, and the structure of quantum group $U_q(\mathfrak{sl}_2)$ *tends to* that of $U(\mathfrak{sl}_2)$ as $q \to 1$. But of course this cannot be regarded as a mathematical treatment at all. So the first goal of this book is to make the above idea rigorous enough to convince ourselves.

In Chapters 1 and 2, we will give a brief review of the basic theory of Lie algebras, Hopf algebras, and Kac-Moody algebras. The notion of *universal enveloping algebras, highest weight modules,* and the *category* $\mathcal{O}_{\text{int}}$ will be introduced. The *Poincaré-Birkhoff-Witt theorem* and the *Weyl-Kac character formula* will be presented without proof. The readers may refer to [**1, 17, 28, 53**] for more detail and complete proofs.

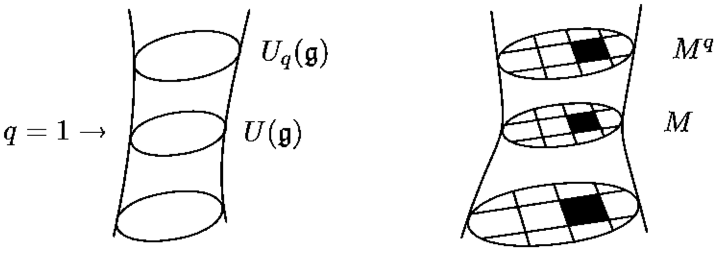
Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra, and let $U(\mathfrak{g})$ be its universal enveloping algebra. In Chapter 3, we will define the *quantum group* $U_q(\mathfrak{g})$ as a certain deformation of $U(\mathfrak{g})$ with a Hopf algebra structure and show that the Hopf algebra structure of $U_q(\mathfrak{g})$ *tends to* that of $U(\mathfrak{g})$ as q approaches 1.

Moreover, we will give a rigorous proof of the statement: *The representation theory of Kac-Moody algebra* $\mathfrak{g}$ *is the same as the representation theory of quantum group* $U_q(\mathfrak{g})$. The essential part of this statement is a theorem proved by G. Lusztig in [**47**]:

The $\mathfrak{g}$*-modules in the category* $\mathcal{O}_{\text{int}}$ (= *integrable modules over* $\mathfrak{g}$ *in the category* $\mathcal{O}$) *can be deformed to* $U_q(\mathfrak{g})$*-modules in the category* $\mathcal{O}^q_{\text{int}}$ *in* *such a way that the dimensions of weight spaces are invariant under the deformation.*

More precisely, let M be a $U(\mathfrak{g})$-module in the category $\mathcal{O}_{\text{int}}$. Then it has a *weight space decomposition* $M = \bigoplus_{\lambda \in P} M_\lambda$, where M_λ is the common eigenspace for the Cartan subalgebra. Now Lusztig's theorem tells that for each generic q, there exists a $U_q(\mathfrak{g})$-module M^q in the category $\mathcal{O}^q_{\text{int}}$ with a weight space decomposition $M^q = \bigoplus_{\lambda \in P} M^q_\lambda$ such that $\dim_{\mathbf{C}(q)} M^q_\lambda = \dim_{\mathbf{C}} M_\lambda$ for all $\lambda \in P$ and that the structure of M^q *tends to* that of M as q approaches 1.

Pictorially, the results obtained in Chapter 3 can be illustrated in the following figure.



Actually, this is one of the motivations for the theory of *crystal bases.* For an integrable module M over $U(\mathfrak{g})$ in the category $\mathcal{O}_{\text{int}}$, consider the formal power series defined by

$$\operatorname{ch} M = \sum_{\lambda \in P} (\dim_{\mathbf{C}} M_\lambda) e^\lambda.$$

The formal series $\operatorname{ch} M$ is called the *character* of the $U(\mathfrak{g})$-module M. The characters of $U(\mathfrak{g})$-modules in the category $\mathcal{O}_{\text{int}}$ *characterize* the representations in the sense that if $M \cong N$, then $\operatorname{ch} M = \operatorname{ch} N$. The converse is not always true, but will hold if the two modules are both highest weight modules with one of them either a Verma module or an irreducible highest weight module. The characters often represent important and interesting mathematical quantities such as *modular forms* in number theory and *one-point functions* in solvable lattice models.

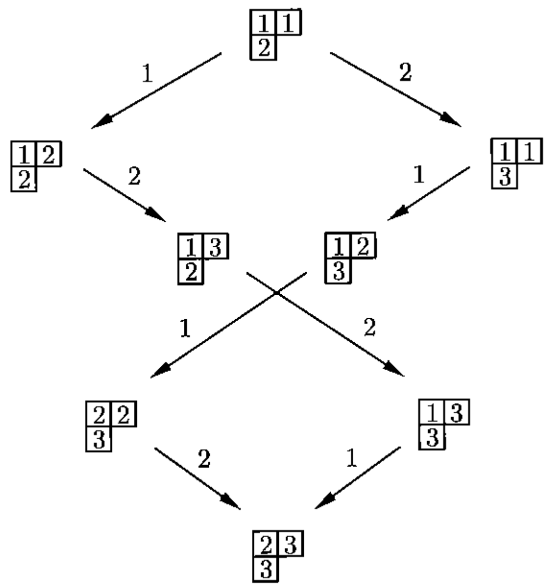
Similarly, one can define the character of a $U_q(\mathfrak{g})$-module M^q in the category $\mathcal{O}^q_{\text{int}}$ to be

$$\operatorname{ch} M^q = \sum_{\lambda \in P} (\dim_{\mathbf{C}(q)} M^q_\lambda) e^\lambda.$$

Since M^q is a quantum deformation of M, by Lusztig's theorem, $\operatorname{ch} M^q$ is the same for all generic parameter q, and it is just the character of M. So if one can calculate $\operatorname{ch} M^q$ for some special value of q, then it suffices to focus on that special case only. The natural question is: *When is the situation simple?* The crystal basis theory tells that it is so when $q = 0$.

In Chapters 4 and 5, we develop the *crystal basis theory* following the combinatorial approach given by Kashiwara [**38, 39**]. In [**48**], a more geometric approach was developed by Lusztig, and it is called the *canonical basis theory*. In [**43–45**], P. Littelmann introduced a combinatorial theory called the *path model* and obtained a colored oriented graph for irreducible highest weight modules over Kac-Moody algebras. It turned out that Littelmann's graphs coincide with Kashiwara's *crystal graphs* ([**25, 40**]).

A *crystal basis* can be understood as a basis at $q = 0$ and is given a structure of colored oriented graph, called the *crystal graph*, with arrows defined by the *Kashiwara operators*. The crystal graphs have many nice combinatorial features reflecting the internal structure of integrable modules over quantum groups. For instance, one of the major goals in combinatorial representation theory is to find an explicit expression for the characters of representations, and this goal can be achieved by finding an explicit combinatorial description of crystal bases. The following picture is the crystal graph for the adjoint representation of $U_q(\mathfrak{sl}_3)$.



Moreover, crystal bases have extremely nice behavior with respect to taking the tensor product. The action of Kashiwara operators is given by the simple *tensor product rule* and the irreducible decomposition of the tensor product of integrable modules is equivalent to decomposing the tensor product of crystal graphs into a disjoint union of connected components. Thus, the crystal basis theory provides us with a powerful combinatorial method of studying the structure of integrable modules over quantum groups.

Our exposition is based on the combinatorial approach developed by Kashiwara [**39**], and some of our arguments overlap with those given in [**21**]. The existence theorem for crystal bases will be proved using Kashiwara's *grand-loop argument* (Section 5.3). We will simplify the original argument, which consists of 14 interlocking inductive statements, to proving 7 interlocking inductive statements. Still, the spirit of the argument is the same as the original one: the fundamental properties of crystal bases for $U_q^-(\mathfrak{g})$ will play the crucial role in the proof.

The next step is to *globalize* the main idea of crystal bases. More precisely, let M^q be a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{\text{int}}$ with crystal basis $(\mathcal{L}, \mathcal{B})$. As we mentioned earlier, the crystal basis $\mathcal{B}$ can be regarded as a *local basis* of M^q at $q = 0$. In Chapter 6, we will show that there exists a unique *global basis* $\mathcal{G}(\mathcal{B}) = \{G(b) \mid b \in \mathcal{B}\}$ of M^q satisfying the properties

$$G(b) \equiv b \mod q\mathcal{L}, \qquad \overline{G(b)} = G(b) \quad \text{for all } b \in \mathcal{B},$$

where $^{-}$ denotes the automorphism on M given by (6.5). The existence theorem for global bases will be proved using the notion of a *balanced triple* and the triviality of vector bundles over $\mathbf{P}^1$. Our argument closely follows the original proof given by M. Kashiwara in [**39**].

Over the past 100 years, it has been discovered that there is a close connection between representation theory and combinatorics. We can see this in the classical works by A. Young ([**57–59**]), D. E. Littlewood and A. R. Richardson ([**46**]), D. Robinson ([**52**]), and H. Weyl ([**55**]). In Chapter 7, we study the connection between the crystal basis theory of finite dimensional $U_q(\mathfrak{gl}_n)$-modules and combinatorics of Young diagrams and Young tableaux. The notion of *admissible reading* (e.g., *Far-Eastern reading* and *Middle-Eastern reading*) lies at the heart of this connection. The crystal graph of a finite dimensional irreducible $U_q(\mathfrak{gl}_n)$-module will be realized as the set of semistandard Young tableaux of a given shape. Moreover, using the tensor product rule for Kashiwara operators, we will give a combinatorial rule (*Littlewood-Richardson rule*) for decomposing the tensor product of finite dimensional $U_q(\mathfrak{gl}_n)$-modules into a direct sum of irreducible components. One may refer to [**46**] for the classical approach.

In Chapter 8, we will extend the above idea to the study of crystal graphs for classical Lie algebras. The crystal graph of a finite dimensional irreducible module over a classical Lie algebra will be realized as the set of semistandard Young tableaux satisfying certain additional conditions depending on the type of the Lie algebra. We will also give a combinatorial rule (*generalized Littlewood-Richardson rule*) for decomposing the tensor product of crystal graphs. Most of the results in Chapters 7 and 8 can be found in [**41**] and [**50**].

As the theory of quantum groups originated from the study of the quantum Yang-Baxter equation, the theory of solvable lattice models can be best explained in the language of representation theory of *quantum affine algebras* (which are the quantum groups corresponding to the affine Kac-Moody algebras). In Chapter 9, we will describe the very basic theory of solvable lattice models and discuss its connection with the representation theory of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ (see, for example, [**24, 36**]). In particular, the *one-point function* for the 6-vertex model will be expressed as the quotient of the string function by the character of the basic representation of $U_q(\widehat{\mathfrak{sl}}_2)$.

In Chapter 10, we will develop the theory of *perfect crystals* for quantum affine algebras (see [**36, 37**]), which has a lot of important applications to the representation theory of quantum affine algebras and vertex models (see, for example, [**7, 24**] and the references therein). We will first study the properties of *vertex operators* and then prove a fundamental crystal isomorphism theorem. Using this crystal isomorphism, the crystal graph of an irreducible highest weight module over a quantum affine algebra will be realized as the set of certain *paths*.

The final chapter will be devoted to the study of crystal bases for basic representations of classical quantum affine algebras using some new combinatorial objects which we call the *Young walls* (see [**34**]). The Young walls consist of colored blocks with various shapes that are built on the given *ground-state wall* and can be viewed as generalizations of Young diagrams. The rules for building Young walls and the action of Kashiwara operators will be given explicitly in terms of combinatorics of Young walls. (They are quite similar to playing with LEGO® blocks and the Tetris® game.) The crystal graph of a basic representation will be characterized as the set of all *reduced proper Young walls.* We expect that there exist interesting and important algebraic structures whose irreducible representations (at some specializations) are parameterized by reduced proper Young walls. It still remains to extend the results in this chapter to the quantum affine algebras of type $C_n^{(1)}$ $(n \geq 3)$.

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NOTATION FROM CH 1 DIFFERING FROM OURS:

$\mathbf{F}$	Arbitrary field of characteristic 0
$\mathbf{Z}$	Ring of integers in $\mathbf{F}$
$\mathbf{Q}$	Field of fractions in $\mathbf{F}$
e, f, h	generators of $\mathfrak{sl}_2(\mathbf{F})$ (our x, y, h)
L, $U(L)$	Lie algebra and its enveloping algebra
$M(\lambda)$	"Verma module", the a highest weight module of weight λ.
Φ, $\Phi_\pm$	Roots, pos/negative roots
r_α	simple reflection associated to α, our s_α or σ_α

Chapter 2

Kac-Moody Algebras

(For notation)

In this chapter, we review the basic theory of Kac-Moody algebras. Our exposition is based on Kac's book ([**28**]). We will omit most of the proofs. Interested readers may refer to [**28, 49**] for more detail.

2.1. Kac-Moody algebras

Let I be a finite index set. A square matrix $A = (a_{ij})_{i,j \in I}$ with entries in $\mathbf{Z}$ is called a ***generalized Cartan matrix*** if it satisfies

$$\begin{aligned} & a_{ii} = 2 \quad \text{for all } i \in I, \\ & a_{ij} \le 0 \quad \text{if } i \ne j, \\ & a_{ij} = 0 \quad \text{if and only if} \quad a_{ji} = 0. \end{aligned} \tag{2.1}$$

If there exists a diagonal matrix $D = \operatorname{diag}(s_i \mid i \in I)$ with all $s_i \in \mathbf{Z}_{>0}$ such that DA is symmetric, then A is said to be ***symmetrizable***. In this book, we will assume that the generalized Cartan matrix A is symmetrizable. The matrix A is ***indecomposable*** if for every pair of nonempty subsets $I_1, I_2 \subset I$ with $I_1 \cup I_2 = I$, there exists some $i \in I_1$ and $j \in I_2$ such that $a_{ij} \ne 0$.

Let $P^\vee$ be a free abelian group of rank $2|I| - \operatorname{rank} A$ with a $\mathbf{Z}$-basis $\{h_i | i \in I\} \cup \{d_s | s = 1, \ldots, |I| - \operatorname{rank} A\}$ and let $\mathfrak{h} = \mathbf{F} \otimes_{\mathbf{Z}} P^\vee$ be the $\mathbf{F}$-linear space spanned by $P^\vee$. We call $P^\vee$ the ***dual weight lattice*** and $\mathfrak{h}$ the ***Cartan subalgebra***. We also define the ***weight lattice*** to be

$$P = \{\lambda \in \mathfrak{h}^* | \lambda(P^\vee) \subset \mathbf{Z}\}.$$

Set $\Pi^\vee = \{h_i | i \in I\}$ and choose a linearly independent subset $\Pi = \{\alpha_i | i \in I\} \subset \mathfrak{h}^*$ satisfying

$$\alpha_j(h_i) = a_{ij}, \qquad \alpha_j(d_s) = 0 \ \text{ or } \ 1 \tag{2.2}$$

for $i, j \in I$, $s = 1, \ldots, |I| - \operatorname{rank} A$. The elements of Π are called ***simple roots***, and the elements of $\Pi^\vee$ are called ***simple coroots***. We also define the ***fundamental weights*** $\Lambda_i \in \mathfrak{h}^*$ $(i \in I)$ to be the linear functionals on $\mathfrak{h}$ given by

$$(2.3) \qquad \Lambda_i(h_j) = \delta_{ij}, \qquad \Lambda_i(d_s) = 0 \quad \text{for } j \in I,\ s = 1, \ldots, |I| - \operatorname{rank} A.$$

Definition 2.1.1. The quintuple $(A, \Pi, \Pi^\vee, P, P^\vee)$ defined as above is said to form a ***Cartan datum*** associated with the generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$.

The free abelian group $Q = \bigoplus_{i\in I} \mathbf{Z}\alpha_i$ is called the ***root lattice*** and $Q_+ = \sum_{i\in I} \mathbf{Z}_{\geq 0}\alpha_i$ is called the ***positive root lattice***. The notation $Q_- = -Q_+$ will also be used. There is a partial ordering on $\mathfrak{h}^*$ defined by $\lambda \geq \mu$ if and only if $\lambda - \mu \in Q_+$ for $\lambda, \mu \in \mathfrak{h}^*$.

For each $i \in I$, we define the ***simple reflection*** r_i on $\mathfrak{h}^*$ by

$$(2.4) \qquad r_i(\lambda) = \lambda - \lambda(h_i)\alpha_i.$$

The subgroup W of $\mathrm{GL}(\mathfrak{h}^*)$ generated by all simple reflections is called the ***Weyl group***. For an element w of the Weyl group, the expression $w = r_{i_1} r_{i_2} \cdots r_{i_t}$ is called a ***reduced expression*** if t is minimal among all such expressions. Note that an element of the Weyl group can have many different reduced expressions. The minimal number t is called the ***length*** of w and is denoted by $l(w)$.

Remark 2.1.2. In [**28**], the triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ is called the *realization* of A. This is sometimes called the *minimal realization*, for $2|I| - \operatorname{rank} A$ is the minimal dimension for the Cartan subalgebra $\mathfrak{h}$ such that α_i $(i \in I)$ are linearly independent. On the other hand, we could use the *maximal realization*. Set $\mathfrak{h}_{\max} = \big(\bigoplus_{i\in I} \mathbf{F}h_i\big) \oplus \big(\bigoplus_{i\in I} \mathbf{F}d_i\big)$ with formal basis elements h_i and d_i $(i \in I)$, and define $\alpha_i \in \mathfrak{h}^*_{\max}$ by setting $\alpha_j(h_i) = a_{ij}$ and $\alpha_j(d_i) = \delta_{ij}$ $(i, j \in I)$. Then the linear functionals α_i are always linearly independent and we can develop the theory of Kac-Moody algebras with the maximal realization in the same way.

We now define the notion of Kac-Moody algebras.

Definition 2.1.3. The ***Kac-Moody algebra*** $\mathfrak{g}$ associated with a Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ is the Lie algebra generated by the elements e_i, f_i $(i \in I)$ and $h \in P^\vee$ subject to the following defining relations:

(1) $[h, h'] = 0$ for $h, h' \in P^\vee$,
(2) $[e_i, f_j] = \delta_{ij} h_i$,
(3) $[h, e_i] = \alpha_i(h) e_i$ for $h \in P^\vee$,
(4) $[h, f_i] = -\alpha_i(h) f_i$ for $h \in P^\vee$,

(5) $(\operatorname{ad} e_i)^{1-a_{i,j}} e_j = 0$ for $i \neq j$,

(6) $(\operatorname{ad} f_i)^{1-a_{i,j}} f_j = 0$ for $i \neq j$.

The relations (1)–(4) are called the *Weyl relations* and the relations (5)–(6) are called the *Serre relations*. We define $\mathfrak{g}_+$ (respectively, $\mathfrak{g}_-$) to be the subalgebra of $\mathfrak{g}$ generated by the elements e_i (respectively, f_i) with $i \in I$, and for each $\alpha \in Q$, let

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \quad \text{for all } h \in \mathfrak{h}\}.$$

The basic properties of Kac-Moody algebras are given in the following proposition.

Proposition 2.1.4. [**28**, Ch.1]

(1) *We have the* ***triangular decomposition***

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+.$$

(2) *We have the* ***root space decomposition***

$$\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha \quad \textit{with } \dim \mathfrak{g}_\alpha < \infty \textit{ for all } \alpha \in Q.$$

(3) *The subalgebra $\mathfrak{g}_+$ (respectively, $\mathfrak{g}_-$) is the Lie algebra generated by the elements e_i $(i \in I)$ (respectively, f_i $(i \in I)$) with the defining relations* (5) (*respectively,* (6)) *in Definition* 2.1.3.

(4) *There exists an involution $\omega : \mathfrak{g} \to \mathfrak{g}$, called the* ***Chevalley involution****, sending $e_i \mapsto -f_i$, $f_i \mapsto -e_i$, and $h \mapsto -h$.*

If $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$, then α is called a ***root*** of $\mathfrak{g}$ and $\mathfrak{g}_\alpha$ is called the ***root space*** attached to α. The dimension of $\mathfrak{g}_\alpha$ is called the ***root multiplicity*** of α. The set of all roots is denoted by Φ. Denote by $\Phi_\pm = \Phi \cap Q_\pm$ the set of positive and negative roots. Every root can be seen to be either positive or negative. The subalgebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is called the ***derived subalgebra***.

The center and the ideals of Kac-Moody algebras are described in the following proposition.

Proposition 2.1.5. [**28**, Ch.1]

(1) *The center of the Kac-Moody algebra $\mathfrak{g}$ is given by*

$$Z(\mathfrak{g}) = \{h \in \mathfrak{h} \mid \alpha_i(h) = 0 \quad \textit{for all } i \in I\}.$$

Hence the dimension of the center is $\dim \mathfrak{h} - |I| = \operatorname{corank} A$.

(2) *Suppose that the generalized Cartan matrix A is indecomposable. Then every ideal of the Kac-Moody algebra $\mathfrak{g}$ either contains its derived subalgebra $\mathfrak{g}'$ or is contained in its center $Z(\mathfrak{g})$.*

We now turn to the *universal enveloping algebra* $U(\mathfrak{g})$ of the Kac-Moody algebra $\mathfrak{g}$. First, note that (Exercise 2.1)

$$(\operatorname{ad} x)^N(y) = \sum_{k=0}^{N} (-1)^k \binom{N}{k} x^{N-k} y x^k \quad \text{for } x, y \in U(\mathfrak{g}),\ N \in \mathbf{Z}_{\geq 0}.$$

Hence we obtain the presentation of $U(\mathfrak{g})$ with generators and relations.

Proposition 2.1.6. *The universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ is the associative algebra over $\mathbf{F}$ with unity generated by e_i, f_i $(i \in I)$ and $\mathfrak{h}$ subject to the following defining relations*:

(1) $hh' = h'h$ *for* $h, h' \in \mathfrak{h}$,
(2) $e_i f_j - f_j e_i = \delta_{ij} h_i$ *for* $i, j \in I$,
(3) $he_i - e_i h = \alpha_i(h) e_i$ *for* $h \in \mathfrak{h}$, $i \in I$,
(4) $hf_i - f_i h = -\alpha_i(h) f_i$ *for* $h \in \mathfrak{h}$, $i \in I$,
(5) $\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} e_i^{1-a_{ij}-k} e_j e_i^k = 0$ *for* $i \neq j$,
(6) $\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} f_i^{1-a_{ij}-k} f_j f_i^k = 0$ *for* $i \neq j$.

As we have seen in Chapter 1, the universal enveloping algebra $U(\mathfrak{g})$ has a Hopf algebra structure given by

$$\begin{aligned} \Delta(x) &= x \otimes 1 + 1 \otimes x, \\ \varepsilon(x) &= 0, \\ S(x) &= -x \quad \text{for } x \in \mathfrak{g}. \end{aligned} \tag{2.5}$$

Let U^+ (respectively, U^0 and U^-) be the subalgebra of $U = U(\mathfrak{g})$ generated by the elements e_i (respectively, $\mathfrak{h}$ and f_i) for $i \in I$. We also define the ***root spaces*** to be

$$\begin{aligned} U_\beta &= \{u \in U \mid hu - uh = \beta(h)u \quad \text{for all } h \in \mathfrak{h}\} \quad \text{for } \beta \in Q, \\ U_\beta^\pm &= \{u \in U^\pm \mid hu - uh = \beta(h)u \quad \text{for all } h \in \mathfrak{h}\} \quad \text{for } \beta \in Q_\pm. \end{aligned}$$

By the Poincaré-Birkhoff-Witt Theorem, the universal enveloping algebra also has the ***triangular decomposition*** and the ***root space decomposition***.

Proposition 2.1.7.

(1) $U(\mathfrak{g}) \cong U^- \otimes U^0 \otimes U^+$.
(2) $U(\mathfrak{g}) = \bigoplus_{\beta \in Q} U_\beta$.
(3) $U^\pm = \bigoplus_{\beta \in Q_\pm} U_\beta^\pm$.

2.2. Classification of generalized Cartan matrices

In this section, we will discuss the classification of generalized Cartan matrices and corresponding Kac-Moody algebras. Let $u = (u_1, \ldots, u_n)^t$ be a column vector in $\mathbf{R}^n$. We say that $u > 0$ (respectively, $u \geq 0$) if $u_i > 0$ (respectively, $u_i \geq 0$) for all $i = 1, \ldots, n$.

Theorem 2.2.1. [**28**, Ch.4] *Let $A = (a_{ij})_{i,j \in I}$ be an indecomposable generalized Cartan matrix. Then one and only one of the following three possibilities hold for both A and A^t.*

(Fin) $\det A \neq 0$; *there exists $u > 0$ such that $Au > 0$; $Av \geq 0$ implies $v > 0$ or $v = 0$.*

(Aff) $\operatorname{corank} A = 1$; *there exists $u > 0$ such that $Au = 0$; $Av \geq 0$ implies $Av = 0$.*

(Ind) *There exists $u > 0$ such that $Au < 0$; $Av \geq 0$ and $v \geq 0$ imply $v = 0$.*

Definition 2.2.2. A generalized Cartan matrix A is said to be of ***finite*** (respectively, ***affine*** or ***indefinite***) ***type*** if A satisfies the condition (Fin) (respectively, (Aff) or (Ind)) in Theorem 2.2.1.

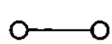
Corollary 2.2.3. [**28**, Ch.4] *An indecomposable generalized Cartan matrix A is of finite (respectively, affine or indefinite) type if there exists $u > 0$ such that $Au > 0$ (respectively, $Au = 0$ or $Au < 0$).*

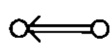
To each generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$, we associate an oriented graph, called the ***Dynkin diagram*** of A. The Dynkin diagram of A consists of vertices indexed by I and edges with arrows defined as follows: If $a_{ij}a_{ji} \leq 4$ and $|a_{ij}| \geq |a_{ji}|$, then the vertices i and j are connected with $|a_{ij}|$ edges equipped with an arrow pointing toward i if $|a_{ij}| > 1$. If $a_{ij}a_{ji} > 4$, then the vertices i and j are connected with a bold-faced edge equipped with an ordered pair of integers $(|a_{ij}|, |a_{ji}|)$.

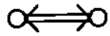
Conversely, from each Dynkin diagram, we can recover the corresponding generalized Cartan matrix, up to the order of indices.

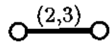
Let us give some examples of 2×2 generalized Cartan matrices and their corresponding Dynkin diagrams.

Example 2.2.4.

(1) $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ 

(2) $A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$ 

(3) $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ 

(4) $A = \begin{pmatrix} 2 & -2 \\ -3 & 2 \end{pmatrix}$ 

A ***subdiagram*** of a Dynkin diagram consists of a subset of the vertices of the original diagram and all edges of the original diagram joining the chosen vertices.

Proposition 2.2.5. [**28**, Ch.4] *Let A be an indecomposable generalized Cartan matrix.*

(1) *A is of finite type if and only if all its principal minors are positive. Equivalently, A is of finite type if and only if all the subdiagrams of the Dynkin diagram of A are of finite type.*

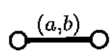
(2) *A is of affine type if and only if* $\det A = 0$ *and all its proper principal minors are positive. Equivalently, A is of affine type if and only if* $\det A = 0$ *and all the proper subdiagrams of the Dynkin diagram of A are of finite type.*

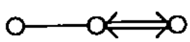
Definition 2.2.6. An indecomposable generalized Cartan matrix A is said to be of ***hyperbolic type*** if A is of indefinite type and every proper subdiagram of the Dynkin diagram of A is of either finite or affine type.

The complete classification of generalized Cartan matrices of finite type and affine type are given in [**28, 54**]. The generalized Cartan matrices of hyperbolic type are classified in [**42, 54**].

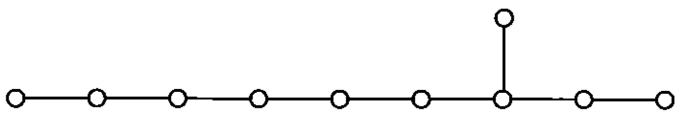
Some examples of generalized Cartan matrices of hyperbolic type and their corresponding Dynkin diagrams are listed below.

Example 2.2.7.

(1) $A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$ with $ab \geq 5$ 

(2) $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$ 

$$(3)\ A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$



One of the fundamental problems in the theory of Kac-Moody algebras is to find an explicit formula for the root multiplicities. For the Kac-Moody algebras of finite type, the root multiplicities are all one. The root multiplicities of affine Kac-Moody algebras are also well known (see, for example, [**28**]). For the Kac-Moody algebras beyond affine type, only limited information is available (see, for example, [**4, 12, 29, 30, 32**]). There do exist formulas for the root multiplicities of Kac-Moody algebras associated with symmetrizable generalized Cartan matrices. In [**5**], S. Berman and R. V. Moody derived a closed form root multiplicity formula and in [**51**] (see also [**28**]), D. Peterson derived a root multiplicity formula in recursive form. In [**32**], using the Euler-Poincaré principle and Kostant's formula for the homology of Kac-Moody algebras, the general root multiplicity formulas were derived both in closed form and in recursive form. However, the behavior of the root multiplicities is still a mystery and no satisfactory description has yet been discovered.

2.3. Representation theory of Kac-Moody algebras

A $\mathfrak{g}$-module V is called a ***weight module*** if it admits a ***weight space decomposition***

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu, \quad \text{where } V_\mu = \{v \in V \mid hv = \mu(h)v \quad \text{for all } h \in \mathfrak{h}\}.$$

A vector $v \in V_\mu$ is called a ***weight vector*** of weight μ. If $e_i v = 0$ for all $i \in I$, v is called a ***maximal vector*** of weight μ. If $V_\mu \neq 0$, μ is called a ***weight*** of V and V_μ is the ***weight space*** attached to μ. Its dimension $\dim V_\mu$ is called the ***weight multiplicity*** of μ. The set of weights of the

$\mathfrak{g}$-module V is denoted by $\mathrm{wt}(V)$. When $\dim V_\mu < \infty$ for all weights μ, the ***character*** of V is defined to be

$$\mathrm{ch}\, V = \sum_{\mu} \dim V_\mu e^\mu,$$

where e^μ are formal basis elements of the group algebra $\mathbf{F}[\mathfrak{h}^*]$ with multiplication defined by $e^\lambda e^\mu = e^{\lambda+\mu}$.

We leave verification of the following proposition to the readers (Exercise 2.4).

Proposition 2.3.1. [**28**, Ch.1] *Every submodule of a weight module is a weight module.*

For $\lambda \in \mathfrak{h}^*$, set $D(\lambda) = \{\mu \in \mathfrak{h}^* \mid \mu \leq \lambda\}$. Let us define the ***category*** $\mathcal{O}$. Its objects consist of weight modules V over $\mathfrak{g}$ with finite dimensional weight spaces for which there exists a finite number of elements $\lambda_1, \lambda_2, \ldots, \lambda_s \in \mathfrak{h}^*$ such that

$$\mathrm{wt}(V) \subset D(\lambda_1) \cup \cdots \cup D(\lambda_s).$$

The morphisms are $\mathfrak{g}$-module homomorphisms. Note that the category $\mathcal{O}$ is closed under taking the finite direct sum or finite tensor product of objects from the category $\mathcal{O}$. Also, the quotients of $\mathfrak{g}$-modules from the category $\mathcal{O}$ are again in the category $\mathcal{O}$.

The most interesting examples of $\mathfrak{g}$-modules in the category $\mathcal{O}$ may be *highest weight modules* given in the following definition.

Definition 2.3.2. A weight module V is a ***highest weight module*** of ***highest weight*** $\lambda \in \mathfrak{h}^*$ if there exists a nonzero vector $v_\lambda \in V$, called a ***highest weight vector***, such that

$$\begin{aligned} &e_i v_\lambda = 0 \quad \text{for all } i \in I, \\ &h v_\lambda = \lambda(h) v_\lambda \quad \text{for all } h \in \mathfrak{h}, \\ &V = U(\mathfrak{g}) v_\lambda. \end{aligned} \tag{2.6}$$

For a highest weight module V, the triangular decomposition of $U = U(\mathfrak{g})$ (Proposition 2.1.7) yields $V = U^- v_\lambda$. Note also that $\dim V_\lambda = 1$, $\dim V_\mu < \infty$ for all $\mu \in \mathrm{wt}(V)$, and $V = \bigoplus_{\mu \leq \lambda} V_\mu$. Thus any highest weight module belongs to the category $\mathcal{O}$ and the name *highest weight module* is justified.

Fix $\lambda \in \mathfrak{h}^*$ and let $J(\lambda)$ be the left ideal of $U(\mathfrak{g})$ generated by all e_i and $h - \lambda(h)1$ ($i \in I$, $h \in \mathfrak{h}$). Set

$$M(\lambda) = U(\mathfrak{g})/J(\lambda).$$

Then $M(\lambda)$ is given a $U(\mathfrak{g})$-module structure by left multiplication. We call $M(\lambda)$ the ***Verma module***. As we have seen in the case of $\mathfrak{sl}_n(\mathbf{F})$-modules, the properties of Verma modules can be summarized in the following proposition.

Proposition 2.3.3. [**28**, Ch.9]

(1) *$M(\lambda)$ is a highest weight $\mathfrak{g}$-module with highest weight λ and highest weight vector $v_\lambda = 1 + J(\lambda)$.*
(2) *Every highest weight $\mathfrak{g}$-module with highest weight λ is a homomorphic image of $M(\lambda)$.*
(3) *As U^--module, $M(\lambda)$ is free of rank 1, generated by the highest weight vector $v_\lambda = 1 + J(\lambda)$.*
(4) *$M(\lambda)$ has a unique maximal submodule.*

Let us denote by $N(\lambda)$ the unique maximal submodule of $M(\lambda)$. The ***irreducible highest weight module*** $M(\lambda)/N(\lambda)$ is denoted by $V(\lambda)$. The importance of irreducible highest weight modules is reflected in the following proposition.

Proposition 2.3.4. [**28**, Ch.9] *Every irreducible $\mathfrak{g}$-module in the category $\mathcal{O}$ is isomorphic to $V(\lambda)$ for some $\lambda \in \mathfrak{h}^*$.*

Let A be a symmetrizable generalized Cartan matrix with a symmetrizing matrix $D = \operatorname{diag}(s_i \mid i \in I)$. Define a symmetric $\mathbf{F}$-valued bilinear form $(\ \mid\ )$ on $\mathfrak{h}$ by

$$(2.7) \qquad \begin{aligned} &(h_i \mid h) = \alpha_i(h)/s_i && \text{for } h \in \mathfrak{h}, \\ &(d_s \mid d_t) = 0 && \text{for } s, t = 1, \dots, |I| - \operatorname{rank} A. \end{aligned}$$

The next lemma may be checked easily (Exercise 2.9).

Lemma 2.3.5. [**28**, Ch.2] *The symmetric bilinear form $(\ \mid\ )$ on $\mathfrak{h}$ is nondegenerate.*

Define a linear map $\nu : \mathfrak{h} \to \mathfrak{h}^*$ by $\nu(h)(h') = (h \mid h')$. The above lemma shows that this map is a vector space isomorphism. Thus we can identify $\mathfrak{h}$ and $\mathfrak{h}^*$ through this map and there is a nondegenerate symmetric bilinear form on $\mathfrak{h}^*$ induced by $(\ \mid\ )$. We will denote this bilinear form by the same notation $(\ \mid\ )$. It satisfies, in particular,

$$(\alpha_i \mid \alpha_j) = s_i a_{ij} \quad \text{for all } i, j \in I.$$

Moreover, it can be easily checked that the symmetric bilinear form $(\ \mid\ )$ is W-invariant; that is, we have (Exercise 2.9)

$$(w\lambda | w\mu) = (\lambda | \mu) \quad \text{for all } w \in W,\ \lambda, \mu \in \mathfrak{h}^*.$$

The nondegenerate symmetric bilinear form on $\mathfrak{h}$ can be extended to a nondegenerate symmetric invariant bilinear form as can be seen in the next proposition.

Proposition 2.3.6. [**28**, Ch.2] *There exists a symmetric bilinear form* $(\,|\,)$ *on* $\mathfrak{g}$ *such that*

(1) $(\,|\,)$ *is given by equations* (2.7) *when restricted to* $\mathfrak{h}$,
(2) $([x,y]\,|\,z) = (x\,|\,[y,z])$ *for all* $x,y,z \in \mathfrak{g}$,
(3) $(\mathfrak{g}_\alpha\,|\,\mathfrak{g}_\beta) = 0$ *if* $\alpha+\beta \neq 0$,
(4) $(\,|\,)$ *is nondegenerate on* $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}$,
(5) $[x,y] = (x\,|\,y)\nu^{-1}(\alpha)$ *for* $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$.

Choose a linear functional $\rho \in \mathfrak{h}^*$ such that $\rho(h_i) = 1$ for all $i \in I$. Let $\{u_i\}$ and $\{u^i\}$ be two bases of $\mathfrak{h}$, dual to each other with respect to $(\,|\,)$. Also, for each positive root α, fix a basis $\{e_\alpha^{(i)}\}$ of $\mathfrak{g}_\alpha$ and $\{f_\alpha^{(i)}\}$ of $\mathfrak{g}_{-\alpha}$ which are dual to each other with respect to $(\,|\,)$. We define the ***Casimir operator*** to be the formal sum

$$\Omega = 2\nu^{-1}(\rho) + \sum_i u_i u^i + 2\sum_{\alpha\in\Phi_+}\sum_i f_\alpha^{(i)} e_\alpha^{(i)}. \tag{2.8}$$

For now, this may be understood as just a formal sum, but it will be a well defined operator on *restricted* $\mathfrak{g}$-modules defined below.

A $\mathfrak{g}$-module V is ***restricted*** if for every $v \in V$, $\mathfrak{g}_\alpha v = 0$ for all but finitely many positive roots α. Thus the action of Casimir operator is well defined on any restricted $\mathfrak{g}$-module.

Proposition 2.3.7. [**28**, Ch.2]

(1) *The action of Casimir operator* Ω *commutes with the action of* $\mathfrak{g}$ *on any restricted* $\mathfrak{g}$*-module* V.
(2) *If* $v \in V$ *is a maximal vector of weight* λ; *i.e., if* $e_i v = 0$ *for every* $i \in I$ *and* $hv = \lambda(h)v$ *for* $h \in \mathfrak{h}$, *then* $\Omega(v) = (\lambda + 2\rho\,|\,\lambda)v$.

Hence, the Casimir operator acts on any highest weight module of highest weight λ by the constant $(\lambda+2\rho\,|\,\lambda)$.

2.4. The category $\mathcal{O}_{\mathrm{int}}$

Let L be a Lie algebra and V an L-module. We say that $x \in L$ is ***locally nilpotent*** on V if for any $v \in V$ there exists a positive integer N such that $x^N \cdot v = 0$.

Lemma 2.4.1. *Let* L *be a Lie algebra and* V *an* L*-module.*

(1) *Let* $\{y_i|\ i \in \Lambda\}$ *be a set of generators of* L *and let* $x \in L$. *If for each* $i \in \Lambda$ *there exists a positive integer* N_i *such that* $(\operatorname{ad} x)^{N_i}(y_i) = 0$, *then* $\operatorname{ad} x$ *is locally nilpotent on* L.

(2) *Let $\{v_i \mid i \in \Lambda\}$ be a set of generators of V and let $x \in L$. If for each $i \in \Lambda$ there exists a positive integer N_i such that $x^{N_i} \cdot v_i = 0$ and $\operatorname{ad} x$ is locally nilpotent on L, then x is locally nilpotent on V.*

Proof. For a positive integer N and $x, y, z \in L$, we have

$$(\operatorname{ad} x)^N([y,z]) = \sum_{k=0}^{N} \binom{N}{k} [(\operatorname{ad} x)^k(y), (\operatorname{ad} x)^{N-k}(z)],$$

$$x^N y = \sum_{k=0}^{N} \binom{N}{k} ((\operatorname{ad} x)^k(y)) x^{N-k}.$$

Here the second equation should be understood as an equation in the universal enveloping algebra with $(\operatorname{ad} x)(y) = xy - yx$. Our assertions follow from the above identities by induction. □

A weight module V over a Kac-Moody algebra $\mathfrak{g}$ is called ***integrable*** if all e_i and f_i $(i \in I)$ are locally nilpotent on V.

Definition 2.4.2. The ***category*** $\mathcal{O}_{\text{int}}$ consists of integrable $\mathfrak{g}$-modules in the category $\mathcal{O}$ such that $\operatorname{wt}(V) \subset P$.

By this definition, any $\mathfrak{g}$-module V in the category $\mathcal{O}_{\text{int}}$ has a weight space decomposition

$$V = \bigoplus_{\lambda \in P} V_\lambda, \quad \text{where } V_\lambda = \{v \in V \mid hv = \lambda(h)v \quad \text{for all } h \in P^\vee\}.$$

Fix $i \in I$. We denote by $\mathfrak{g}_{(i)}$ (respectively, $U_{(i)}$) the subalgebra of $\mathfrak{g}$ (respectively, $U(\mathfrak{g})$) generated by e_i, f_i, h_i. Then we have $\mathfrak{g}_{(i)} \cong \mathfrak{sl}_2$ and $U_{(i)} \cong U(\mathfrak{sl}_2)$. Let V be a $\mathfrak{g}$-module in the category $\mathcal{O}_{\text{int}}$. Since e_i and f_i are locally nilpotent on V, there is a well defined $\mathfrak{g}$-module automorphism of V given by

$$\tau_i = (\exp f_i)(\exp(-e_i))(\exp f_i). \tag{2.9}$$

Moreover we can prove:

Proposition 2.4.3. [28, Ch.3] *Let V be a $\mathfrak{g}$-module in the category $\mathcal{O}_{\text{int}}$.*

(1) *For each $i \in I$, V decomposes into a direct sum of finite dimensional irreducible $\mathfrak{h}$-invariant $\mathfrak{g}_{(i)}$-submodules.*

(2) *We have*

$$\tau_i V_\lambda = V_{r_i \lambda} \quad \textit{for all } i \in I,\ \lambda \in \operatorname{wt}(V).$$

Hence $\dim V_\lambda = \dim V_{w\lambda}$ for all $w \in W$, $\lambda \in \operatorname{wt}(V)$.

By Lemma 2.4.1, a highest weight $\mathfrak{g}$-module with highest weight λ and highest weight vector v_λ is integrable if and only if for every $i \in I$, there exists $N_i \in \mathbf{Z}_{\geq 0}$ such that $f_i^{N_i} v_\lambda = 0$.

Define the set of ***dominant integral weights*** to be

$$P^+ = \{\lambda \in P \mid \lambda(h_i) \in \mathbf{Z}_{\geq 0} \quad \text{for all } i \in I\}.$$

Lemma 2.4.4. [28, Ch.10]

(1) *Let $V(\lambda)$ be the irreducible highest weight $\mathfrak{g}$-module with highest weight $\lambda \in \mathfrak{h}^*$. Then $V(\lambda)$ lies in the category $\mathcal{O}_{\text{int}}$ if and only if $\lambda \in P^+$.*

(2) *Every irreducible $\mathfrak{g}$-module in the category $\mathcal{O}_{\text{int}}$ is isomorphic to $V(\lambda)$ for some $\lambda \in P^+$.*

Proof. (1) Suppose $V(\lambda)$ lies in the category $\mathcal{O}_{\text{int}}$ and let v_λ be a highest weight vector of $V(\lambda)$. Then, by definition, $\lambda \in P$, and for each $i \in I$, there exists a nonnegative integer N_i such that $f_i^{N_i} \cdot v_\lambda \neq 0$ and $f_i^{N_i+1} v_\lambda = 0$. Thus we have

$$0 = e_i f_i^{N_i+1} v_\lambda = (N_i + 1)(\lambda(h_i) - (N_i + 1) + 1) f_i^{N_i} v_\lambda,$$

which implies $\lambda(h_i) = N_i \in \mathbf{Z}_{\geq 0}$. Hence $\lambda \in P^+$.

Conversely, if $\lambda \in P^+$, consider the vector $f_i^{\lambda(h_i)+1} v_\lambda$. If $j \neq i$, then $e_j f_i^{\lambda(h_i)+1} v_\lambda = 0$. Moreover, we have

$$e_i f_i^{\lambda(h_i)+1} v_\lambda = (\lambda(h_i) + 1)(\lambda(h_i) - (\lambda(h_i) + 1) + 1) f_i^{\lambda(h_i)} v_\lambda = 0.$$

Hence if $f_i^{\lambda(h_i)+1} v_\lambda \neq 0$, since its weight is $\lambda - (\lambda(h_i) + 1)\alpha_i < \lambda$, it would generate a nontrivial proper submodule of V, which contradicts the irreducibility of $V(\lambda)$. Therefore, $f_i^{\lambda(h_i)+1} v_\lambda = 0$ for all $i \in I$ and hence $V(\lambda)$ is integrable. Clearly, $\text{wt}(V) \subset P$, and $V(\lambda)$ lies in the category $\mathcal{O}_{\text{int}}$.

(2) By Proposition 2.3.4, every irreducible $\mathfrak{g}$-module V in the category $\mathcal{O}$ is isomorphic to $V(\lambda)$ for some $\lambda \in \mathfrak{h}^*$. If V lies in $\mathcal{O}_{\text{int}}$, by (1), we must have $\lambda \in P^+$. □

Remark 2.4.5. We have just seen that if $V(\lambda)$ is an irreducible highest weight $\mathfrak{g}$-module with highest weight $\lambda \in P^+$ and a highest weight vector v_λ, then we have $f_i^{\lambda(h_i)+1} v_\lambda = 0$ for all $i \in I$. Actually, as we can see in the following theorem, the converse is also true: if V is a highest weight $\mathfrak{g}$-module with highest weight $\lambda \in P^+$ and highest weight vector v_λ such that $f_i^{\lambda(h_i)+1} v_\lambda = 0$ for all $i \in I$, then V is isomorphic to the irreducible highest weight $\mathfrak{g}$-module $V(\lambda)$.

Theorem 2.4.6. [**28**, Ch.10], [**49**, Ch.6] *Let $\mathfrak{g}$ be a Kac-Moody algebra associated with a Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$, and let V be a highest weight $\mathfrak{g}$-module with highest weight $\lambda \in P^+$ and highest weight vector v_λ.*

If $f_i^{\lambda(h_i)+1} v_\lambda = 0$ for all $i \in I$, then the character of V is given by

$$\operatorname{ch} V = \frac{\sum_{w \in W} (-1)^{l(w)} e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha \in \Phi_+} (1 - e^{-\alpha})^{\dim \mathfrak{g}_\alpha}}. \tag{2.10}$$

In particular, V is isomorphic to the irreducible highest weight $\mathfrak{g}$-module $V(\lambda)$.

The formula (2.10) is called the ***Weyl-Kac character formula***.

Corollary 2.4.7. [**28**, Ch.10] *Every highest weight $\mathfrak{g}$-module in the category $\mathcal{O}_{\text{int}}$ is isomorphic to some $V(\lambda)$ with $\lambda \in P^+$.*

Proof. Let V be a highest weight $\mathfrak{g}$-module in the category $\mathcal{O}_{\text{int}}$ with highest weight λ and highest weight vector v_λ. From the first part of the proof for Lemma 2.4.4 (1), we find that the nonnegative integer N_i satisfying $f_i^{N_i} \cdot v_\lambda \neq 0$ and $f_i^{N_i+1} v_\lambda = 0$ is actually $\lambda(h_i)$. So we have $\lambda \in P^+$ and $f_i^{\lambda(h_i)+1} v_\lambda = 0$ for all $i \in I$. Hence, by the Weyl-Kac character formula, we obtain $V \cong V(\lambda)$. □

Letting $\lambda = 0$ in (2.10), we obtain the ***denominator identity***

$$\prod_{\alpha \in \Phi_+} (1 - e^{-\alpha})^{\dim \mathfrak{g}_\alpha} = \sum_{w \in W} (-1)^{l(w)} e^{w\rho - \rho}. \tag{2.11}$$

The denominator identity is a rich source of interesting mathematical research activity. For instance, the root multiplicity formulas for Kac-Moody algebras mentioned in Section 2.2 were all derived from the denominator identity. Moreover, when it is applied to the affine Kac-Moody algebra of type $A_1^{(1)}$ associated with the generalized Cartan matrix $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, it yields the famous *Jacobi triple product identity* ([**26, 28**]):

$$\prod_{n=1}^{\infty} (1 - p^n q^n)(1 - p^{n-1} q^n)(1 - p^n q^{n-1}) = \sum_{k \in \mathbf{Z}} (-1)^k p^{\frac{k(k-1)}{2}} q^{\frac{k(k+1)}{2}}.$$

The denominator identity can also be interpreted as the Euler-Poincaré principle for the Kac-Moody algebras. (See [**31**], [**32**] and [**33**] for more detail and further developments in this direction.)

We conclude this section with a complete reducibility theorem for $\mathfrak{g}$-modules in the category $\mathcal{O}_{\text{int}}$.

Theorem 2.4.8. [28, Ch.10] *Let $\mathfrak{g}$ be a Kac-Moody algebra associated with a Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$. Then every $\mathfrak{g}$-module in the category $\mathcal{O}_{\text{int}}$ is isomorphic to a direct sum of irreducible highest weight modules $V(\lambda)$ with $\lambda \in P^+$.*

We will not give a proof for this theorem. But a quantum version of this theorem will be proved in Section 3.5. The original proof given by Kac [27] for the nonquantum case uses properties of the Casimir operator. The proof for the quantum case, which does not use Casimir operator, may easily be adopted to the nonquantum case.

Corollary 2.4.9. [28, Ch.10] *The tensor product of a finite number of $\mathfrak{g}$-modules in the category $\mathcal{O}_{\text{int}}$ is completely reducible.*

Exercises

2.1. Let L be a Lie algebra and $U(L)$ be its universal enveloping algebra. Verify that for any $x, y \in U(L)$ and $N \in \mathbf{Z}_{\geq 0}$, we have

$$(\operatorname{ad} x)^N(y) = \sum_{k=0}^{k} (-1)^k \binom{N}{k} x^{N-k} y x^k.$$

2.2. Classify all the Dynkin diagrams of affine type with n vertices containing the Dynkin diagram A_{n-1} as a subdiagram.

2.3. Let W be the Weyl group of a Cartan datum. Show that

$$l(w) = |\{\alpha \in \Phi_+ \,|\, w\alpha < 0\}|.$$

2.4. Prove that every submodule of a weight module is also a weight module.

2.5. (a) Show that the center of a Kac-Moody algebra $\mathfrak{g}$ is

$$Z(\mathfrak{g}) = \{h \in \mathfrak{h} \,|\, \alpha_i(h) = 0 \quad \text{for all } i \in I\}.$$

(b) Show that $Z(\mathfrak{g}) \subset \mathfrak{h}' = \bigoplus_{i \in I} \mathbf{F} h_i$ and that $\dim Z(\mathfrak{g}) = \operatorname{corank} A$.

2.6. Let $\mathfrak{g}$ be a Kac-Moody algebra associated with an indecomposable generalized Cartan matrix. Prove that every ideal of $\mathfrak{g}$ either contains $\mathfrak{g}'$ or is contained in $Z(\mathfrak{g})$.

2.7. Verify the properties of the Verma module stated in Proposition 2.3.3.

2.8. Show that every irreducible $\mathfrak{g}$-module in the category $\mathcal{O}$ is isomorphic to $V(\lambda)$ for some $\lambda \in \mathfrak{h}^*$.

2.9. Verify that the symmetric bilinear form $(\ |\ )$ on $\mathfrak{h}$ defined by (2.7) is nondegenerate on $\mathfrak{h}$ and is W-invariant.

2.10. Let V be a $\mathfrak{g}$-module in the category $\mathcal{O}_{\text{int}}$ and let Ω be the Casimir operator on V. Show that if v is a maximal vector of weight λ, then

$$\Omega(v) = (\lambda + 2\rho|\lambda)v.$$

2.11. Verify the properties of the automorphism τ_i of $\mathfrak{g}$ given in Proposition 2.4.3.

2.12. Let $\mathfrak{g}$ be the affine Kac-Moody algebra of type $A_2^{(2)}$ associated with the generalized Cartan matrix $A = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$. Show that the denominator identity yields the *quintuple product identity*

$$\prod_{n=1}^{\infty}(1-p^{2n}q^n)(1-p^{2n-1}q^{n-1})(1-p^{2n-1}q^n)(1-p^{4n-4}q^{2n-1})(1-p^{4n}q^{2n-1})$$
$$= \sum_{k\in\mathbf{Z}} \left(p^{3k^2-2k}q^{(3k^2+k)/2} - p^{3k^2-4k+1}q^{(3k^2-k)/2}\right).$$

Hint: The root system of $\mathfrak{g}$ is given by

$$\Phi^{\text{re}} = \{(2n\pm 1)\alpha_0 + n\alpha_1,\ 4n\alpha_0 + (2n\pm 1)\alpha_1 \mid n \in \mathbf{Z}\},$$
$$\Phi^{\text{im}} = \{2n\alpha_0 + n\alpha_1 \mid n \in \mathbf{Z}, n \neq 0\},$$

where $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Phi$. (See [**28**, Ch.8].)

Chapter 3

Quantum Groups

In this chapter, we introduce the *quantum deformations* of the universal enveloping algebras of Kac-Moody algebras, or in more popular terms, the *quantum groups* $U_q(\mathfrak{g})$. We will show that many of the features of the universal enveloping algebras of Kac-Moody algebras carry over to the quantum groups and that the quantum groups are true *deformations* of the universal enveloping algebras. We will also show that the representation theory of Kac-Moody algebras can be *deformed* to the representation theory of quantum groups.

3.1. Quantum groups

In this section, we construct the *quantum deformation* $U_q(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ of a Kac-Moody algebra $\mathfrak{g}$. It will be given a noncommutative, noncocommutative Hopf algebra structure and we will show that it admits a triangular decomposition. The base field $\mathbf{F}$ will be, as before, an arbitrary field of characteristic zero.

Given $n \in \mathbf{Z}$ and any symbol x, we define the notation

$$[n]_x = \frac{x^n - x^{-n}}{x - x^{-1}}. \tag{3.1}$$

We define $[0]_x! = 1$ and $[n]_x! = [n]_x[n-1]_x \cdots [1]_x$ for $n \in \mathbf{Z}_{>0}$.

For nonnegative integers $m \geq n \geq 0$, the analogues of binomial coefficients are given by

$$\begin{bmatrix} m \\ n \end{bmatrix}_x = \frac{[m]_x!}{[n]_x![m-n]_x!}. \tag{3.2}$$

Fix an indeterminate q. Then, $[n]_q$ and $\begin{bmatrix} m \\ n \end{bmatrix}_q$ are elements of the field $\mathbf{F}(q)$, which are called ***q-integers*** and ***q-binomial coefficients***, respectively. We may show inductively, using the identity

$$\begin{bmatrix} m+1 \\ n \end{bmatrix}_q = q^n \begin{bmatrix} m \\ n \end{bmatrix}_q + q^{-m+n-1} \begin{bmatrix} m \\ n-1 \end{bmatrix}_q, \tag{3.3}$$

that these elements actually belong to $\mathbf{Z}[q, q^{-1}]$ (Exercise 3.1). Note that we have

$$[n]_q \to n \quad \text{and} \quad \begin{bmatrix} m \\ n \end{bmatrix}_q \to \binom{m}{n} \quad \text{as} \quad q \to 1.$$

Let $A = (a_{ij})_{i,j \in I}$ be a symmetrizable generalized Cartan matrix with a symmetrizing matrix $D = \operatorname{diag}(s_i \in \mathbf{Z}_{>0} \mid i \in I)$ and let $(A, \Pi, \Pi^\vee, P, P^\vee)$ be a Cartan datum associated with A.

Definition 3.1.1. The ***quantum group*** or the ***quantized universal enveloping algebra*** $U_q(\mathfrak{g})$ associated with a Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ is the associative algebra over $\mathbf{F}(q)$ with 1 generated by the elements e_i, f_i ($i \in I$) and q^h ($h \in P^\vee$) with the following defining relations:

(1) $q^0 = 1$, $q^h q^{h'} = q^{h+h'}$ for $h, h' \in P^\vee$,

(2) $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i$ for $h \in P^\vee$,

(3) $q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i$ for $h \in P^\vee$,

(4) $e_i f_j - f_j e_i = \delta_{ij} \dfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$ for $i, j \in I$,

(5) $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} e_i^{1-a_{ij}-k} e_j e_i^k = 0$ for $i \neq j$,

(6) $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} f_i^{1-a_{ij}-k} f_j f_i^k = 0$ for $i \neq j$.

Here, $q_i = q^{s_i}$ and $K_i = q^{s_i h_i}$. For $\alpha = \sum_i n_i \alpha_i \in Q$, the notation $K_\alpha = \prod_i K_i^{n_i}$ will also be used.

Set $\deg f_i = -\alpha_i$, $\deg q^h = 0$, and $\deg e_i = \alpha_i$. Since all the defining relations of the quantum group $U_q(\mathfrak{g})$ are homogeneous, it has a ***root space decomposition***

$$U_q(\mathfrak{g}) = \bigoplus_{\alpha \in Q} (U_q)_\alpha, \tag{3.4}$$

where $(U_q)_\alpha = \{u \in U_q(\mathfrak{g}) \mid q^h u q^{-h} = q^{\alpha(h)} u \text{ for all } h \in P^\vee\}$.

The last two defining relations above are called the *quantum Serre relations*. Define

$$(\mathrm{ad}_q\, x)(y) = xy - q^{(\alpha|\beta)}yx \quad \text{for} \quad x \in (U_q)_\alpha, y \in (U_q)_\beta \quad (\alpha, \beta \in Q),$$

and extend it to all of $U_q(\mathfrak{g})$ by linearity. These are called *quantum adjoint operators*. Then, we get (Exercise 3.2)

$$(\mathrm{ad}_q\, e_i)^N(e_j) = \sum_{k=0}^{N}(-1)^k q_i^{k(N+a_{ij}-1)} \begin{bmatrix} N \\ k \end{bmatrix}_{q_i} e_i^{N-k} e_j e_i^k.$$

Hence the quantum Serre relations may be written in the form

$$(3.5) \qquad (\mathrm{ad}_q\, e_i)^{1-a_{ij}}(e_j) = 0, \qquad (\mathrm{ad}_q\, f_i)^{1-a_{ij}}(f_j) = 0 \quad \text{for } i \neq j.$$

We will now show that the quantum group $U_q(\mathfrak{g})$ has a Hopf algebra structure.

Proposition 3.1.2. *The quantum group $U_q(\mathfrak{g})$ has a Hopf algebra structure with the comultiplication Δ, counit ε, and antipode S defined by*

(1) $\Delta(q^h) = q^h \otimes q^h$,
(2) $\Delta(e_i) = e_i \otimes K_i^{-1} + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + K_i \otimes f_i$,
(3) $\varepsilon(q^h) = 1, \quad \varepsilon(e_i) = \varepsilon(f_i) = 0$,
(4) $S(q^h) = q^{-h}, \quad S(e_i) = -e_iK_i, \quad S(f_i) = -K_i^{-1}f_i$

for $h \in P^\vee$ and $i \in I$.

Proof. The maps have been defined only on the generators. So we first extend them to the free associative algebra on the given generators by requiring Δ and ε to be algebra homomorphisms and by requiring S to be an antihomomorphism of algebras. To show that these maps are well defined, it suffices to show that all the defining relations are preserved under these maps. The first four relations in Definition 3.1.1 can be easily verified.

To prove that the antipode preserves the quantum Serre relations, we use

$$S(e_i^{N-k}e_je_i^k) = (-1)^{N+1}q_i^{N(N+a_{ij}-1)}e_i^ke_je_i^{N-k}K_i^NK_j,$$
$$S(f_i^{N-k}f_jf_i^k) = (-1)^{N+1}q_i^{-N(N+a_{ij}-1)}K_i^{-N}K_j^{-1}f_i^kf_jf_i^{N-k},$$

both of which may be obtained by using the first three defining relations.

We will now prove that comultiplication preserves the quantum Serre relations. By induction, we can show

$$\begin{aligned}\Delta((\mathrm{ad}_q\, e_i)^N(e_j)) &= (\mathrm{ad}_q\, e_i)^N(e_j)\otimes K_i^{-N}K_j^{-1}\\ &+\sum_{k=0}^{N-1}\tau_k^{(N)}q_i^{k(N-k)}\begin{bmatrix}N\\k\end{bmatrix}_{q_i} e_i^{N-k}\otimes K_i^{-N+k}(\mathrm{ad}_q\, e_i)^k(e_j)\\ &+1\otimes(\mathrm{ad}_q\, e_i)^N(e_j),\end{aligned}$$

where $\tau_k^{(N)}=\prod_{t=k}^{N-1}(1-q_i^{2(t+a_{ij})})$ (Exercise 3.2). Setting $N=1-a_{ij}$, the middle term vanishes and we see that comultiplication preserves the quantum Serre relations.

It remains to check if these maps actually satisfy the conditions for Hopf algebras given in Definition 1.5.3. We have only to verify that these conditions are satisfied on the generators of $U_q(\mathfrak{g})$, which is straightforward (Exercise 3.3). □

Let U_q^+ (respectively, U_q^-) be the subalgebra of $U_q(\mathfrak{g})$ generated by the elements e_i (respectively, f_i) for $i\in I$, and let U_q^0 be the subalgebra of $U_q(\mathfrak{g})$ generated by q^h ($h\in P^\vee$). In addition, let $U_q^{\geq 0}$ (respectively, $U_q^{\leq 0}$) be the subalgebra of $U_q(\mathfrak{g})$ generated by e_i ($i\in I$) and q^h ($h\in P^\vee$) (resp. f_i ($i\in I$) and q^h ($h\in P^\vee$)). We would like to show that the quantum group $U_q(\mathfrak{g})$ has the ***triangular decomposition***

$$U_q(\mathfrak{g})\cong U_q^-\otimes U_q^0\otimes U_q^+.$$

To do this, we first introduce an involution on $U_q(\mathfrak{g})$. Define a linear map $T: U_q(\mathfrak{g})\to U_q(\mathfrak{g})$ by

$$T(q^h)=q^{-h},\quad T(e_i)=f_i,\quad T(f_i)=e_i\quad (h\in P^\vee, i\in I).$$

It is easy to verify that T defines an algebra endomorphism on $U_q(\mathfrak{g})$. Let $\sigma: U_q(\mathfrak{g})\otimes U_q(\mathfrak{g})\to U_q(\mathfrak{g})\otimes U_q(\mathfrak{g})$ be the transposition map defined by

$$\sigma(a\otimes b)=b\otimes a\quad \text{for } a,b\in U_q(\mathfrak{g}).$$

As the following proposition shows, T is actually an involution.

Proposition 3.1.3.

(1) $T^2=\mathrm{id}$.

(2) $\Delta\circ T=\sigma\circ(T\otimes T)\circ\Delta$.

(3) *T restricted to U_q^+ gives an algebra isomorphism between U_q^+ and U_q^-.*

Proof. It suffices to check them on the generators, which is quite straightforward. □

The following lemma is the key step in proving triangular decomposition.

Lemma 3.1.4.

(1) $U_q^{\geq 0} \cong U_q^0 \otimes U_q^+$.

(2) $U_q^{\leq 0} \cong U_q^- \otimes U_q^0$.

Proof. We will just prove the second part. Let $\{f_\zeta\}_{\zeta\in\Omega}$ be a basis of U_q^- consisting of monomials in f_i's ($i \in I$). Consider the map $\varphi : U_q^- \otimes U_q^0 \to U_q^{\leq 0}$ given by $\varphi(f_\zeta \otimes q^h) = f_\zeta q^h$. Since

$$q^h f_\zeta = q^{-\beta(h)} f_\zeta q^h \quad \text{for } f_\zeta \in (U_q^-)_{-\beta},\ \beta \in Q_+,$$

φ is surjective. Thus it is enough to show that $\{f_\zeta q^h \mid \zeta \in \Omega, h \in P^\vee\}$ is linearly independent over $\mathbf{F}(q)$.

Suppose

$$\sum_{\substack{\zeta\in\Omega \\ h\in P^\vee}} C_{\zeta,h} f_\zeta q^h = 0 \quad \text{for some } C_{\zeta,h} \in \mathbf{F}(q).$$

We may write

$$\sum_{\beta\in Q_+} \Big(\sum_{\substack{\deg f_\zeta = -\beta \\ h\in P^\vee}} C_{\zeta,h} f_\zeta q^h \Big) = 0.$$

(Here, we denote $\deg u = \beta \in Q$ if $u \in (U_q)_\beta$.)

Since $U_q = \bigoplus_{\beta\in Q} (U_q)_\beta$, we have

$$\sum_{\substack{\deg f_\zeta = -\beta \\ h\in P^\vee}} C_{\zeta,h} f_\zeta q^h = 0 \quad \text{for each } \beta \in Q_+. \tag{3.6}$$

Since each f_ζ is a monomial in f_i's ($i \in I$), if it is of degree $-\beta \in Q_-$, we have

$$\Delta(f_\zeta) = f_\zeta \otimes 1 + (\text{intermediate terms}) + K_\beta \otimes f_\zeta.$$

Applying the comultiplication Δ to (3.6) yields

$$\sum_{\substack{\deg f_\zeta = -\beta \\ h\in P^\vee}} C_{\zeta,h} \big(f_\zeta q^h \otimes q^h + \cdots + K_\beta q^h \otimes f_\zeta q^h \big) = 0.$$

Collecting the terms of degree $(-\beta,0)$, we obtain

$$\sum_{\substack{\deg f_\zeta = -\beta \\ h\in P^\vee}} C_{\zeta,h} \big(f_\zeta q^h \otimes q^h \big) = 0.$$

Since the set $\{q^h\}_{h\in P^\vee}$ is linearly independent, we have

$$\sum_{\deg f_\zeta = -\beta} C_{\zeta,h} f_\zeta q^h = 0 \quad \text{for all } h \in P^\vee.$$

Multiplying by q^{-h} from the right and using linear independence of f_ζ, we conclude all $C_{\zeta,h} = 0$ as desired. □

We are now ready to prove the *triangular decomposition* for $U_q(\mathfrak{g})$.

Theorem 3.1.5. $U_q(\mathfrak{g}) \cong U_q^- \otimes U_q^0 \otimes U_q^+$.

Proof. Let $\{f_\zeta\}_{\zeta\in\Omega}$ and $\{e_\zeta\}_{\zeta\in\Omega}$ be monomial bases of U_q^- and U_q^+, respectively. As in the proof for Lemma 3.1.4, it suffices to show that the set $\{f_\zeta q^h e_\eta \,|\, \zeta,\eta \in \Omega, h \in P^\vee\}$ is linearly independent over $\mathbf{F}(q)$.

Suppose

$$\sum_{\zeta,h,\eta} C_{\zeta,h,\eta} f_\zeta q^h e_\eta = 0 \quad \text{for some } C_{\zeta,h,\eta} \in \mathbf{F}(q).$$

The root space decomposition of $U_q(\mathfrak{g})$ shows that

$$\sum_{\substack{h\in P^\vee \\ \deg f_\zeta + \deg e_\eta = \gamma}} C_{\zeta,h,\eta} f_\zeta q^h e_\eta = 0 \quad \text{for all } \gamma \in Q.$$

We know

$$\Delta(e_\eta) = e_\eta \otimes K^{-1}_{\deg e_\eta} + \cdots + 1 \otimes e_\eta,$$
$$\Delta(f_\zeta) = f_\zeta \otimes 1 + \cdots + K_{-\deg f_\zeta} \otimes f_\zeta.$$

Therefore,

$$\begin{aligned} 0 &= \sum_{\substack{h\in P^\vee \\ \deg f_\zeta + \deg e_\eta = \gamma}} C_{\zeta,h,\eta} \Delta\left(f_\zeta q^h e_\eta\right) \\ &= \sum_{\substack{h\in P^\vee \\ \deg f_\zeta + \deg e_\eta = \gamma}} C_{\zeta,h,\eta} (f_\zeta \otimes 1 + \cdots)(q^h \otimes q^h)(\cdots + 1 \otimes e_\eta). \end{aligned} \tag{3.7}$$

Recall the partial ordering on $\mathfrak{h}^*$ defined in Section 2.1 and choose $\alpha = \deg f_\zeta$ and $\beta = \deg e_\eta$, which are minimal and maximal, respectively, among those for which $C_{\zeta,h,\eta}$ is nonzero. The terms in (3.7) of degree (α,β) must sum to zero. Hence,

$$\sum_{\substack{h\in P^\vee, \\ \deg f_\zeta = \alpha, \deg e_\eta = \beta}} C_{\zeta,h,\eta} (f_\zeta q^h \otimes q^h e_\eta) = 0.$$

Since the vectors $f_\zeta q^h$ are linearly independent by Lemma 3.1.4, we have

$$\sum_{\deg e_\eta = \beta} C_{\zeta,h,\eta} q^h e_\eta = 0 \quad \text{for all } \zeta \text{ and } h.$$

From this, we may conclude that $C_{\zeta,h,\eta} = 0$, as desired. □

3.2. Representation theory of quantum groups

In this section, we study representations of the quantum group. The theory is quite parallel to that of Kac-Moody algebras.

A $U_q(\mathfrak{g})$-module V^q is called a ***weight module*** if it admits a ***weight space decomposition***

$$V^q = \bigoplus_{\mu \in P} V^q_\mu, \quad \text{where } V^q_\mu = \{v \in V^q \mid q^h v = q^{\mu(h)} v \quad \text{for all } h \in P^\vee\}.$$

A vector $v \in V^q_\mu$ is called a ***weight vector*** of weight μ. If $e_i v = 0$ for all $i \in I$, it is called a ***maximal vector***. If $V^q_\mu \neq 0$, μ is called a ***weight*** of V^q and V^q_μ is the ***weight space*** attached to $\mu \in P$. Its dimension $\dim V^q_\mu$ is called the ***weight multiplicity*** of μ. We will denote by $\mathrm{wt}(V^q)$ the set of weights of the $U_q(\mathfrak{g})$-module V^q. If $\dim V^q_\mu < \infty$ for all $\mu \in \mathrm{wt}(V^q)$, the ***character*** of V^q is defined by

$$\mathrm{ch}\, V^q = \sum_\mu \dim V^q_\mu e^\mu,$$

where e^μ are formal basis elements of the group algebra $\mathbf{F}[P]$ with multiplication defined by $e^\lambda e^\mu = e^{\lambda+\mu}$.

Proposition 3.2.1. *Every submodule of a weight module over $U_q(\mathfrak{g})$ is also a weight module.*

Proof. Let V^q be a weight module. Suppose there exists some submodule W^q which is not a weight module. Choose $v = v_1 + \cdots + v_p \in W^q$, where $v_k \in V^q_{\mu_k}$, μ_k are distinct, and $v_k \notin W^q$ for some k. We may assume further that every element of W^q with fewer summands has all its summands belonging to W^q. This forces $v_k \notin W^q$ for all k. Choose any $h \in P^\vee$ such that $\mu_1(h) \neq \mu_k(h)$ for at least one k. Then $q^h v - q^{\mu_1(h)} v$ is a nonzero element of W^q with a strictly smaller number of summands for which all its summands do not belong to W^q, which is a contradiction. □

For $\lambda \in P$, set $D(\lambda) = \{\mu \in P \mid \mu \leq \lambda\}$. The ***category*** $\mathcal{O}^q$ consists of weight modules V^q over $U_q(\mathfrak{g})$ with finite dimensional weight spaces for which there exist a finite number of elements $\lambda_1, \lambda_2, \ldots, \lambda_s \in P$ such that

$$\mathrm{wt}(V^q) \subset D(\lambda_1) \cup \cdots \cup D(\lambda_s).$$

As is the case with Kac-Moody algebras, the most important examples among the $U_q(\mathfrak{g})$-modules in the category $\mathcal{O}^q$ may be *highest weight modules*. A weight module V^q is called a ***highest weight module*** with ***highest***

weight $\lambda \in P$ if there exists a nonzero $v_\lambda \in V^q$ such that

$$\begin{aligned} &e_i v_\lambda = 0 \quad \text{for all } i \in I, \\ &q^h v_\lambda = q^{\lambda(h)} v_\lambda \quad \text{for all } h \in P^\vee, \\ &V^q = U_q(\mathfrak{g}) v_\lambda. \end{aligned} \tag{3.8}$$

The vector v_λ, which is unique up to constant multiple, is called the ***highest weight vector***. The triangular decomposition for $U_q(\mathfrak{g})$ (Theorem 3.1.5) shows $V^q = U_q^- v_\lambda$ for any highest weight module. It can be easily verified that $\dim V^q_\lambda = 1$, $\dim V^q_\mu < \infty$ for all $\mu \in \mathrm{wt}(V^q)$, and $V^q = \bigoplus_{\mu \le \lambda} V^q_\mu$. The last property justifies the name *highest weight modules.*

Fix $\lambda \in P$ and let $J^q(\lambda)$ be the left ideal of $U_q(\mathfrak{g})$ generated by e_i $(i \in I)$ and $q^h - q^{\lambda(h)}1$ $(h \in P^\vee)$. Define the ***Verma module*** $M^q(\lambda) = U_q(\mathfrak{g})/J^q(\lambda)$. This is a $U_q(\mathfrak{g})$-module by left multiplication. Set $v_\lambda = 1 + J^q(\lambda)$. Then we have

$$\begin{aligned} &q^h v_\lambda = q^h + J^q(\lambda) = q^{\lambda(h)}1 + J^q(\lambda) = q^{\lambda(h)} v_\lambda, \\ &e_i v_\lambda = e_i + J^q(\lambda) = J^q(\lambda) = 0, \\ &U_q(\mathfrak{g}) v_\lambda = U_q(\mathfrak{g})/J^q(\lambda) = M^q(\lambda). \end{aligned}$$

Thus $M^q(\lambda)$ is a highest weight module with highest weight λ and highest weight vector $v_\lambda = 1 + J^q(\lambda)$.

Proposition 3.2.2.

(1) *As a U_q^--module, $M^q(\lambda)$ is free of rank 1, generated by the highest weight vector $v_\lambda = 1 + J^q(\lambda)$.*

(2) *Every highest weight $U_q(\mathfrak{g})$-module with highest weight λ is a homomorphic image of $M^q(\lambda)$.*

(3) *The Verma module $M^q(\lambda)$ has a unique maximal submodule.*

Proof. (1) Any highest weight module is generated by its highest weight vector as a U_q^--module, so it only remains to prove that it is free. We need to prove that $u v_\lambda = 0$ for $u \in U_q^-$ implies $u = 0$ or, equivalently, $U_q^- \cap J^q(\lambda) = 0$. Combining Lemma 3.1.4 and the triangular decomposition (Theorem 3.1.5), we may write $U_q(\mathfrak{g}) \cong U_q^- \otimes U_q^{\ge 0}$. Thus $J^q(\lambda)$, the left ideal of $U_q(\mathfrak{g})$ generated by e_i $(i \in I)$ and $q^h - q^{\lambda(h)}1$ $(h \in P^\vee)$, cannot have elements that lie in U_q^-.

(2) Let W^q be an arbitrary highest weight module with highest weight λ and highest weight vector w_λ. Then (1) allows us to define a map $\phi : M^q(\lambda) \to W^q$ by $u \cdot (1 + J^q(\lambda)) \mapsto u \cdot w_\lambda$ for $u \in U_q^-$. Since $W^q = U_q^- w_\lambda$, the map ϕ is surjective. It remains to check if ϕ is a $U_q(\mathfrak{g})$-module homomorphism.

The action of an arbitrary $x \in U_q(\mathfrak{g})$ on an arbitrary $uv_\lambda \in M^q(\lambda)$ with $u \in U_q(\mathfrak{g})^-$ may be computed as follows. First, write $xu \in U_q(\mathfrak{g})$ in the form given by the triangular decomposition, say, $xu = \sum u^- u^0 u^+$ with $u^\pm \in U_q^\pm$ and $u^0 \in U_q^0$. We want to see if xuv_λ is sent to xuw_λ by the map ϕ. This is equivalent to checking if $\sum u^- u^0 u^+ v_\lambda$ is sent to $\sum u^- u^0 u^+ w_\lambda$. Since the actions of q^h and e_i on v_λ and w_λ are identical and result in only constant multiples of v_λ, each $u^0 u^+ v_\lambda$ is sent to the corresponding $u^0 u^+ w_\lambda$. The remaining action of u^- is preserved by construction of the map. Hence the map ϕ defined above does preserve $U_q(\mathfrak{g})$-action.

(3) Note that any proper submodule of $M^q(\lambda)$ cannot contain the highest weight vector $v_\lambda = 1 + J^q(\lambda)$; that is, it must lie inside $\bigoplus_{\mu \lneq \lambda} M^q(\lambda)_\mu$. Hence the sum of two proper submodules is again a proper submodule of $M^q(\lambda)$. Therefore, the sum of all proper submodules of $M^q(\lambda)$ is the unique maximal submodule of $M^q(\lambda)$. □

We denote this unique maximal submodule of $M^q(\lambda)$ by $N^q(\lambda)$. Then the quotient $M^q(\lambda)/N^q(\lambda)$ is an ***irreducible highest weight module*** with highest weight λ, which will be denoted by $V^q(\lambda)$.

We now define the main object of our study in this chapter—the *category* $\mathcal{O}_{\text{int}}^q$ of $U_q(\mathfrak{g})$-modules. A weight module V^q over the quantum group $U_q(\mathfrak{g})$ is ***integrable*** if all e_i and f_i $(i \in I)$ are locally nilpotent on V^q.

Definition 3.2.3. The ***category*** $\mathcal{O}_{\text{int}}^q$ consists of $U_q(\mathfrak{g})$-modules V^q satisfying the following conditions:

(1) V^q has a weight space decomposition $V^q = \bigoplus_{\lambda \in P} V_\lambda^q$, where

$$V_\lambda^q = \{v \in V^q \mid q^h v = q^{\lambda(h)} v \ \text{ for all } h \in P^\vee\}$$

and $\dim V_\lambda^q < \infty$ for all $\lambda \in P$,

(2) there exist a finite number of elements $\lambda_1, \ldots, \lambda_s \in P$ such that

$$\text{wt}(V^q) \subset D(\lambda_1) \cup \cdots \cup D(\lambda_s),$$

(3) all e_i and f_i $(i \in I)$ are locally nilpotent on V^q.

The morphisms are taken to be usual $U_q(\mathfrak{g})$-module homomorphisms.

Hence the category $\mathcal{O}_{\text{int}}^q$ consists of integrable $U_q(\mathfrak{g})$-modules in the category $\mathcal{O}^q$. Note that the category $\mathcal{O}_{\text{int}}^q$ is closed under taking direct sums or tensor products of finitely many $U_q(\mathfrak{g})$-modules.

Fix $i \in I$. We denote by $U_q(\mathfrak{g}_{(i)})$ the subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i, $K_i^{\pm 1}$. Then we have $U_q(\mathfrak{g}_{(i)}) \cong U_{q_i}(\mathfrak{sl}_2)$, and, as for the $\mathfrak{g}$-modules in the category $\mathcal{O}_{\text{int}}$, we can prove (Exercise 3.5):

Proposition 3.2.4 ([21, 28]). *Let V^q be a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{\text{int}}$. Then, for each $i \in I$, V^q decomposes into a direct sum of $U_q(\mathfrak{h})$-invariant finite dimensional irreducible $U_q(\mathfrak{g}_{(i)})$-submodules.*

Set $e_i^{(k)} = e_i^k/[k]_{q_i}!$ and $f_i^{(k)} = f_i^k/[k]_{q_i}!$. They are called the ***divided powers*** of e_i and f_i, respectively. We have the following commutation relations for the divided powers, which can be proved by straightforward induction (Exercise 3.6).

Lemma 3.2.5. *For all $i \in I$ and $k \in \mathbf{Z}_{\geq 0}$, we have*

$$e_i f_i^{(k)} = f_i^{(k)} e_i + f_i^{(k-1)} \frac{K_i q_i^{-k+1} - K_i^{-1} q_i^{k-1}}{q_i - q_i^{-1}}.$$

Proposition 3.2.6. *Let $\lambda \in P^+$ and let $V^q(\lambda)$ be the irreducible highest weight module of highest weight λ and highest weight vector v_λ. Then $f_i^{\lambda(h_i)+1} v_\lambda = 0$ for all $i \in I$.*

Proof. The above lemma shows

$$e_i f_i^{(k)} v_\lambda = [\lambda(h_i) - k + 1]_{q_i} f_i^{(k-1)} v_\lambda. \tag{3.9}$$

Substituting $k = \lambda(h_i) + 1$, we see that $e_i f_i^{\lambda(h_i)+1} v_\lambda = 0$. Moreover, for $j \neq i$, we already know $e_j f_i^{\lambda(h_i)+1} v_\lambda = 0$. Hence if $f_i^{\lambda(h_i)+1} v_\lambda \neq 0$, it would generate a nontrivial proper submodule of $V^q(\lambda)$, contradicting the irreducibility of $V^q(\lambda)$. □

Proposition 3.2.7. *A highest weight $U_q(\mathfrak{g})$-module V^q with highest weight $\lambda \in P$ and highest weight vector v_λ is integrable if and only if for every $i \in I$, there exists some N_i such that $f_i^{N_i} v_\lambda = 0$.*

Proof. We have only to prove the *if* part. Note that $e_i \cdot V_\mu \subset V_{\mu+\alpha_i}$. Since all the weights of a highest weight module are less than or equal to its highest weight, the e_i ($i \in I$) are always locally nilpotent on any highest weight module. So we restrict our attention to the f_i's only.

For homogeneous $u \in U_q^-$ of degree $-\alpha \in Q_-$, we have

$$f_i^n u = \sum_{k=0}^{n} q_i^{(\alpha(h_i)+k)(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i} \left((\mathrm{ad}_q f_i)^k (u) \right) f_i^{n-k}.$$

With the help of (3.3), we may check this by induction (Exercise 3.7). Recall that $(\mathrm{ad}_q f_i)^k (f_j) = 0$ if $j \neq i$ and $k > -a_{i,j}$. So if $u = f_j u'$ with $j \neq i$, we have

$$f_i^n u = \sum_{k=0}^{-a_{i,j}} q_i^{(\alpha(h_i)+k)(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i} \left((\mathrm{ad}_q f_i)^k (f_j) \right) f_i^{n-k} u'.$$

If $u = f_i u'$, we may set $f_i^n u = f_i^{n+1} u'$. Given $u \in U_q^-$, this shows how we may inductively prove $f_i^n u \in U_q^- \cdot f_i^{N_i}$ for all sufficiently large n. Now, an arbitrary element of V^q may be written in the form $u \cdot v_\lambda$ with $u \in U_q^-$, which completes the proof. □

Proposition 3.2.8. *Let $V^q(\lambda)$ be the irreducible highest weight $U_q(\mathfrak{g})$-module with highest weight $\lambda \in P$. Then $V^q(\lambda)$ belongs to the category $\mathcal{O}_{\mathrm{int}}^q$ if and only if $\lambda \in P^+$.*

Proof. The *if* part is taken care of by Propositions 3.2.6 and 3.2.7. Let us prove the *only if* part. Fix $i \in I$ and let N_i be the smallest nonnegative integer such that $f_i^{N_i} v_\lambda \neq 0$ and $f_i^{N_i+1} v_\lambda = 0$. Then we have

$$0 = e_i f_i^{(N_i+1)} v_\lambda = [\lambda(h_i) - N_i]_{q_i} f_i^{(N_i)} v_\lambda,$$

which implies

$$[\lambda(h_i) - N_i]_{q_i} = \frac{q_i^{\lambda(h_i)-N_i} - q_i^{-\lambda(h_i)+N_i}}{q_i - q_i^{-1}} = 0.$$

It follows that $q_i^{2(\lambda(h_i)-N_i)} = 1$. Since q is an indeterminate, we must have $\lambda(h_i) = N_i \in \mathbf{Z}_{\geq 0}$. □

Remark 3.2.9. As in Lemma 2.4.4 (2), we would like to claim that every irreducible $U_q(\mathfrak{g})$-module in the category $\mathcal{O}_{\mathrm{int}}^q$ is isomorphic to the irreducible highest weight module $V^q(\lambda)$ for some $\lambda \in P^+$. For this, we need to wait until the end of the first half of Section 3.4. In Section 3.4, we will also show that every highest weight $U_q(\mathfrak{g})$-module in the category $\mathcal{O}_{\mathrm{int}}^q$ is isomorphic to $V^q(\lambda)$ with $\lambda \in P^+$.

3.3. $\mathbf{A}_1$-forms

In the previous section, we have seen that the representation theory of $U_q(\mathfrak{g})$ is very similar to that of $\mathfrak{g}$. Hence it is natural to expect that the quantum group $U_q(\mathfrak{g})$ may be regarded as some sort of *deformation* of $U(\mathfrak{g})$ in such a way that the representations of $U_q(\mathfrak{g})$ can also be regarded as the *deformations* of those of $U(\mathfrak{g})$. Moreover, from the defining relations, we can expect the structures of the quantum group $U_q(\mathfrak{g})$ and its representations *tend to* those of $U(\mathfrak{g})$ and its representations as q approaches 1. This observation is one of the most fundamental properties of quantum groups and their representations which was first proved in [**47**].

In this and the next section, we make precise and prove these statements. By somewhat modifying Lusztig's approach, we show that the quantum group $U_q(\mathfrak{g})$ is a *deformation* of $U(\mathfrak{g})$ as a Hopf algebra and show that a highest weight $U(\mathfrak{g})$-module admits a *deformation* to a highest weight $U_q(\mathfrak{g})$-module in such a way that the dimensions of the weight spaces remain the same under the deformation.

We consider the localization of $\mathbf{F}[q]$ at the ideal $(q-1)$:

$$\begin{aligned}\mathbf{A}_1 &= \{f(q) \in \mathbf{F}(q) \mid f \text{ is regular at } q=1\} \\ &= \{g/h \mid g, h \in \mathbf{F}[q],\ h(1) \neq 0\}.\end{aligned} \tag{3.10}$$

Notice that $[n]_{q_i} \in \mathbf{A}_1$ and $\begin{bmatrix} m \\ n \end{bmatrix}_{q_i} \in \mathbf{A}_1$, being elements of $\mathbf{Z}[q, q^{-1}]$. For an integer $n \in \mathbf{Z}$, we formally define

$$[y; n]_x = \frac{yx^n - y^{-1}x^{-n}}{x - x^{-1}} \quad \text{and} \quad (y; n)_x = \frac{yx^n - 1}{x - 1}. \tag{3.11}$$

For example, we have

$$\begin{aligned}[q_i^m; n]_{q_i} &= \frac{q_i^{m+n} - q_i^{-m-n}}{q_i - q_i^{-1}} \in \mathbf{A}_1, \\ (q_i^m; n)_{q_i} &= \frac{q_i^{m+n} - 1}{q_i - 1} \in \mathbf{A}_1, \\ [q^h; n]_q &= \frac{q^h q^n - q^{-h} q^{-n}}{q - q^{-1}} \in U_q^0, \\ (q^h; n)_q &= \frac{q^h q^n - 1}{q - 1} \in U_q^0.\end{aligned}$$

Definition 3.3.1. We define the $\mathbf{A}_1$***-form***, denoted by $U_{\mathbf{A}_1}$, of the quantum group $U_q(\mathfrak{g})$ to be the $\mathbf{A}_1$-subalgebra of $U_q(\mathfrak{g})$ generated by the elements e_i, f_i, q^h, and $(q^h; 0)_q$ ($i \in I$, $h \in P^\vee$).

Let $U_{\mathbf{A}_1}^+$ (respectively, $U_{\mathbf{A}_1}^-$) be the $\mathbf{A}_1$-subalgebra of $U_{\mathbf{A}_1}$ generated by the elements e_i (respectively, f_i) for $i \in I$, and let $U_{\mathbf{A}_1}^0$ be the $\mathbf{A}_1$-subalgebra of $U_{\mathbf{A}_1}$ generated by q^h and $(q^h; 0)_q$ for $h \in P^\vee$. The next lemma shows that $U_{\mathbf{A}_1}^0$ contains all of the more frequently appearing elements of U_q^0.

Lemma 3.3.2.

(1) $(q^h; n)_q \in U_{\mathbf{A}_1}^0$ *for all* $n \in \mathbf{Z}$ *and* $h \in P^\vee$.

(2) $[K_i; n]_{q_i} \in U_{\mathbf{A}_1}^0$ *for all* $n \in \mathbf{Z}$ *and* $i \in I$.

Proof. It suffices to check the following identities:

$$(q^h; n)_q = q^n (q^h; 0)_q + \frac{q^n - 1}{q - 1},$$
$$[K_i; 0]_{q_i} = q_i \frac{q - 1}{q_i^2 - 1}(1 + K_i^{-1})(K_i; 0)_q,$$
$$[K_i; n]_{q_i} = q_i^n [K_i; 0]_{q_i} + [n]_{q_i} K_i^{-1},$$

all of which can be verified by straightforward calculations (Exercise 3.8). □

We next show that the triangular decomposition of $U_q(\mathfrak{g})$ carries over to its $\mathbf{A}_1$-form.

Proposition 3.3.3. *We have a natural isomorphism of* $\mathbf{A}_1$*-modules*

$$U_{\mathbf{A}_1} \cong U_{\mathbf{A}_1}^- \otimes U_{\mathbf{A}_1}^0 \otimes U_{\mathbf{A}_1}^+$$

induced from the triangular decomposition of $U_q(\mathfrak{g})$.

Proof. Consider the canonical isomorphism $\varphi : U_q(\mathfrak{g}) \xrightarrow{\sim} U_q^- \otimes U_q^0 \otimes U_q^+$ given by Theorem 3.1.5. The commutation relations

$$e_i (q^h; 0)_q = (q^h; -\alpha_i(h))_q e_i,$$
$$(q^h; 0)_q f_i = f_i (q^h; -\alpha_i(h))_q,$$
$$e_i f_j = f_j e_i + \delta_{i,j} [K_i; 0]_{q_i},$$

together with Lemma 3.3.2 show that the image of φ lies inside $U_{\mathbf{A}_1}^- \otimes U_{\mathbf{A}_1}^0 \otimes U_{\mathbf{A}_1}^+$ when restricted to $U_{\mathbf{A}_1}$. Its inverse map is given by multiplication. Hence the two spaces are isomorphic as $\mathbf{A}_1$-modules. □

Fix $\lambda \in P$. Throughout this and the next section, V^q will denote a highest weight $U_q(\mathfrak{g})$-module with highest weight λ and highest weight vector v_λ.

Definition 3.3.4. The $\mathbf{A}_1$***-form*** of the highest weight module V^q with highest weight $\lambda \in P$ and highest weight vector v_λ is defined to be the $U_{\mathbf{A}_1}$-module $V_{\mathbf{A}_1} = U_{\mathbf{A}_1} v_\lambda$.

First, observe that we have:

Proposition 3.3.5. $V_{\mathbf{A}_1} = U_{\mathbf{A}_1}^- v_\lambda$.

Proof. In view of Proposition 3.3.3, it suffices to show that $U_{\mathbf{A}_1}^+ v_\lambda = \mathbf{A}_1 v_\lambda$ and $U_{\mathbf{A}_1}^0 v_\lambda = \mathbf{A}_1 v_\lambda$. The first assertion is clear by the definition of highest weight modules. For the second assertion, we observe that

$$q^h v_\lambda = q^{\lambda(h)} v_\lambda,$$
$$(q^h;0)_q v_\lambda = \frac{q^{\lambda(h)}-1}{q-1} v_\lambda.$$

Hence we get $U_{\mathbf{A}_1} v_\lambda = U^-_{\mathbf{A}_1} v_\lambda$. □

Recall that the highest weight $U_q(\mathfrak{g})$-module V^q has the weight space decomposition

$$V^q = \bigoplus_{\mu \le \lambda} V^q_\mu, \quad \text{where } V^q_\mu = \{v \in V^q \mid q^h v = q^{\mu(h)} v \quad \text{for all } h \in P^\vee\}.$$

For each $\mu \in P$, define the ***weight space*** $(V_{\mathbf{A}_1})_\mu = V_{\mathbf{A}_1} \cap V^q_\mu$. The next proposition shows that the weight space decomposition of V^q also carries over to its $\mathbf{A}_1$-form.

Proposition 3.3.6. $V_{\mathbf{A}_1} = \bigoplus_{\mu \le \lambda} (V_{\mathbf{A}_1})_\mu$.

Proof. Assume $v = v_1 + \cdots + v_p \in V_{\mathbf{A}_1}$, where $v_j \in V^q_{\mu_j}$ and $\mu_j \in P$. We may take μ_j to be distinct. Fix an index j. It suffices to show $v_j \in V_{\mathbf{A}_1}$.

For each $k \neq j$, we may choose $H_k \in P^\vee$ such that $\mu_j(H_k) \neq \mu_k(H_k)$. Note that

$$(q^{\mu_j(H_k)}; -\mu_k(H_k))_q = \frac{q^{\mu_j(H_k)-\mu_k(H_k)}-1}{q-1}$$

is invertible in $\mathbf{A}_1$ for each $k \neq j$. Imitating *Lagrange's interpolation formula*, define $u \in U_{\mathbf{A}_1}$ to be

$$u = \prod_{k \neq j} \frac{(q^{H_k}; -\mu_k(H_k))_q}{(q^{\mu_j(H_k)}; -\mu_k(H_k))_q}.$$

Then $uv_j = v_j$ and $uv_k = 0$ for $k \neq j$. Hence $uv = v_j \in V_{\mathbf{A}_1}$. □

An approach to proving the above proposition that mimics the proof of Proposition 3.2.1 fails because the scalars we are dealing with do not form a field. But the proof for the next proposition relies heavily on the fact that $\mathbf{A}_1$ is close enough to a field.

Proposition 3.3.7. *For each* $\mu \in P$, *the weight space* $(V_{\mathbf{A}_1})_\mu$ *is a free* $\mathbf{A}_1$*-module with* $\operatorname{rank}_{\mathbf{A}_1} (V_{\mathbf{A}_1})_\mu = \dim_{\mathbf{F}(q)} V^q_\mu$.

Proof. Notice that $(V_{\mathbf{A}_1})_\mu$ is finitely generated as an $\mathbf{A}_1$-module. Let $\{v_k\}_{k=1}^p$ be an $\mathbf{A}_1$-spanning set of $(V_{\mathbf{A}_1})_\mu$. We will show that this spanning set can be reduced to an $\mathbf{A}_1$-linearly independent set. Then we would have an $\mathbf{A}_1$-basis of $(V_{\mathbf{A}_1})_\mu$, which would imply $(V_{\mathbf{A}_1})_\mu$ is a free $\mathbf{A}_1$-module. Consider an arbitrary $\mathbf{A}_1$-linear dependence relation

$$c_1(q)v_1 + \cdots + c_p(q)v_p = 0 \tag{3.12}$$

with each $c_k(q) \in \mathbf{A}_1$. Dividing out by $(q-1)$ if necessary, we may assume that at least one of the coefficients satisfies $c_k(1) \neq 0$. For example, suppose $c_1(1) \neq 0$. Then $c_1(q)^{-1} \in \mathbf{A}_1$ and

$$v_1 = \frac{-1}{c_1(q)}\{c_2(q)v_2 + \cdots + c_p(q)v_p\}.$$

Repeating this process, we get an $\mathbf{A}_1$-linearly independent spanning set of $(V_{\mathbf{A}_1})_\mu$.

As for its rank, let $\{f_\zeta v_\lambda\}$ be a basis of V_μ^q, where f_ζ are monomials in f_i. The set certainly belongs to $(V_{\mathbf{A}_1})_\mu$ and is also $\mathbf{A}_1$-linearly independent, so $\operatorname{rank}_{\mathbf{A}_1}(V_{\mathbf{A}_1})_\mu \geq \dim_{\mathbf{F}(q)} V_\mu^q$. To show the converse inequality, let $\{v_k\}_{k=1}^p$ be an $\mathbf{A}_1$-linearly independent subset of $(V_{\mathbf{A}_1})_\mu$. Consider an $\mathbf{F}(q)$-linear dependence relation

$$b_1(q)v_1 + \cdots + b_p(q)v_p = 0,$$

where $b_k(q) \in \mathbf{F}(q)$ for $k = 1, \ldots, p$. Multiplying by $(q-1)$ if needed, we may assume that all $b_k(q) \in \mathbf{A}_1$. Since $v_1, \ldots, v_p$ are linearly independent over $\mathbf{A}_1$, we must have $b_k(q) = 0$ for all $k = 1, \ldots, p$. Hence $v_1, \ldots, v_p$ are linearly independent over $\mathbf{F}(q)$ and $\operatorname{rank}_{\mathbf{A}_1}(V_{\mathbf{A}_1})_\mu \leq \dim_{\mathbf{F}(q)} V_\mu^q$, which completes the proof. □

Proposition 3.3.8. *The $\mathbf{F}(q)$-linear map $\varphi : \mathbf{F}(q) \otimes_{\mathbf{A}_1} V_{\mathbf{A}_1} \to V^q$ given by $c \otimes v \mapsto cv$ $(c \in \mathbf{F}(q),\ v \in V_{\mathbf{A}_1})$ is an isomorphism.*

Proof. Combining Propositions 3.3.6 and 3.3.7, we get the desired linear isomorphism. □

Remark 3.3.9.

(1) We see from this proposition that the $\mathbf{A}_1$-form $V_{\mathbf{A}_1}$ of a highest weight module V^q is an *integral form* of V^q over $\mathbf{A}_1$; i.e., it can be viewed as an $\mathbf{A}_1$-lattice in V^q.

(2) In Exercise 3.10, we give an alternative proof of Proposition 3.3.7 which works for more general setting.

3.4. Classical limit

We now proceed to take the limit $q \to 1$ of highest weight $U_q(\mathfrak{g})$-modules. The notation from the previous section will be retained. In particular, V^q

will denote a highest weight $U_q(\mathfrak{g})$-module of highest weight $\lambda \in P$ and highest weight vector v_λ. Let $\mathbf{J}_1$ be the unique maximal ideal of the local ring $\mathbf{A}_1$ generated by $q-1$. There exists an isomorphism of fields

$$\mathbf{A}_1/\mathbf{J}_1 \xrightarrow{\sim} \mathbf{F} \quad \text{given by} \quad f(q) + \mathbf{J}_1 \mapsto f(1).$$

(In particular, q is mapped onto 1.) Define the $\mathbf{F}$-linear vector spaces

$$\begin{aligned} U_1 &= (\mathbf{A}_1/\mathbf{J}_1) \otimes_{\mathbf{A}_1} U_{\mathbf{A}_1}, \\ V^1 &= (\mathbf{A}_1/\mathbf{J}_1) \otimes_{\mathbf{A}_1} V_{\mathbf{A}_1}. \end{aligned} \tag{3.13}$$

Then V^1 is naturally a U_1-module. We would like to show that U_1 is isomorphic to the universal enveloping algebra $U(\mathfrak{g})$ and that V^1 is a highest weight $U(\mathfrak{g})$-module of highest weight λ.

Note that

$$U_1 \cong U_{\mathbf{A}_1}/\mathbf{J}_1 U_{\mathbf{A}_1} \quad \text{and} \quad V^1 \cong V_{\mathbf{A}_1}/\mathbf{J}_1 V_{\mathbf{A}_1}.$$

Consider the natural maps

$$\begin{aligned} U_{\mathbf{A}_1} &\to U_{\mathbf{A}_1}/\mathbf{J}_1 U_{\mathbf{A}_1} \cong U_1, \\ V_{\mathbf{A}_1} &\to V_{\mathbf{A}_1}/\mathbf{J}_1 V_{\mathbf{A}_1} \cong V^1. \end{aligned} \tag{3.14}$$

We use the bar notation for the image under these maps. The passage under these maps is referred to as taking the ***classical limit***. Notice that q is mapped to 1 under these maps. The notation U_1 has been used to call to mind "$U_q(\mathfrak{g})$ at $q = 1$".

For each $\mu \in P$, define $V^1_\mu = (\mathbf{A}_1/\mathbf{J}_1) \otimes_{\mathbf{A}_1} (V_{\mathbf{A}_1})_\mu$. Then we have:

Lemma 3.4.1.

(1) *For each $\mu \in P$, if $\{v_i\}$ is a basis of the free $\mathbf{A}_1$-module $(V_{\mathbf{A}_1})_\mu$, then $\{\bar{v}_i\}$ is a basis of the $\mathbf{F}$-linear space V^1_μ.*

(2) *For each $\mu \in P$, a set $\{v_i\} \subset (V_{\mathbf{A}_1})_\mu$ is $\mathbf{A}_1$-linearly independent if the set $\{\bar{v}_i\} \subset V^1_\mu$ is $\mathbf{F}$-linearly independent.*

Proof. (1) Using [**18**, Thm.5.11, Ch.4], we may show that $\{1 \otimes v_i\}$ is a basis of the $(\mathbf{A}_1/\mathbf{J}_1)$-linear space V^1_μ (Exercise 3.11).

(2) Suppose $\sum c_i(q) v_i = 0$ for $c_i(q) \in \mathbf{A}_1$, not all zero. Dividing out by $q-1$, if necessary, we may assume at least one $c_i(1) \neq 0$. Then, $\sum c_i(1)\bar{v}_i = 0$ is a nontrivial $\mathbf{F}$-linear dependence relation. □

Proposition 3.4.2.

(1) $V^1 = \bigoplus_{\mu \le \lambda} V^1_\mu$.

(2) *For each $\mu \in P$,* $\dim_{\mathbf{F}} V^1_\mu = \operatorname{rank}_{\mathbf{A}_1} (V_{\mathbf{A}_1})_\mu$.

Proof. The first follows from Proposition 3.3.6. And the second follows from Lemma 3.4.1. □

We now know

$$\dim_{\mathbf{F}} V^1_\mu = \operatorname{rank}_{\mathbf{A}_1}(V_{\mathbf{A}_1})_\mu = \dim_{\mathbf{F}(q)} V^q_\mu \tag{3.15}$$

for all $\mu \in P$.

Let $\bar{h} \in U_1$ denote the classical limit of the element

$$(q^h;0)_q = \frac{q^h - 1}{q-1} \in U_{\mathbf{A}_1}.$$

We first show that the image of $U^0_{\mathbf{A}_1}$ under the classical limit is quite close to $U^0 = U(\mathfrak{h})$.

Lemma 3.4.3.

(1) *For all $h \in P^\vee$, we have $\overline{q^h} = 1$.*
(2) *For any $h, h' \in P^\vee$, $\overline{h+h'} = \bar{h} + \bar{h}'$. Hence, $\overline{nh} = n\bar{h}$ for $n \in \mathbf{Z}$.*

Proof. (1) Note that

$$q^h - 1 = (q-1)(q^h;0)_q.$$

Hence the classical limit of the right-hand side, being a multiple of $q-1$, is zero.

(2) We may easily calculate

$$(q^{h+h'};0)_q = q^{h'}(q^h;0)_q + (q^{h'};0)_q.$$

We take the classical limit of both sides using $\overline{q^{h'}} = 1$ to obtain the desired result. □

Define the subalgebras $U^0_1 = \mathbf{F} \otimes U^0_{\mathbf{A}_1}$ and $U^\pm_1 = \mathbf{F} \otimes U^\pm_{\mathbf{A}_1}$. The next theorem shows that the classical limit of $U_q(\mathfrak{g})$ is almost the same as $U(\mathfrak{g})$.

Theorem 3.4.4.

(1) *The elements $\bar{e}_i$, $\bar{f}_i$, $(i \in I)$ and $\bar{h}$ $(h \in P^\vee)$ satisfy the defining relations of $U(\mathfrak{g})$ given by Proposition 2.1.6. Hence, there exists a surjective $\mathbf{F}$-algebra homomorphism $\psi : U(\mathfrak{g}) \to U_1$ and the U_1-module V^1 has a $U(\mathfrak{g})$-module structure.*

(2) *For each $\mu \in P$ and $h \in P^\vee$, the element $\bar{h}$ acts on V^1_μ as scalar multiplication by $\mu(h)$. So V^1_μ is the μ-weight space of the $U(\mathfrak{g})$-module V^1.*

(3) *As a $U(\mathfrak{g})$-module, V^1 is a highest weight module with highest weight $\lambda \in P$ and highest weight vector $\bar{v}_\lambda$.*

Proof. (1) The first relation for $U(\mathfrak{g})$ is trivial. Let us check the second defining relation. By definition of $U_q(\mathfrak{g})$, we have

$$e_i f_i - f_i e_i = [K_i; 0]_{q_i} = \frac{q_i}{q_i+1}\frac{q-1}{q_i-1}(1+K_i^{-1})(K_i;0)_q.$$

Taking Lemma 3.4.3 into account, the classical limit of the right-hand side is

$$\frac{1}{2}\cdot\frac{1}{s_i}\cdot 2\cdot s_i\bar{h}_i = \bar{h}_i,$$

which yields the second defining relation. As for the third defining relation, note that

$$\begin{aligned}(q^h;0)_q e_i - e_i(q^h;0)_q &= e_i(q^h;\alpha_i(h))_q - e_i(q^h;0)_q \\ &= \frac{q^{\alpha_i(h)}-1}{q-1}e_i q^h.\end{aligned}$$

We take the classical limit of both sides to obtain $\bar{h}\bar{e}_i - \bar{e}_i\bar{h} = \alpha_i(h)\bar{e}_i$. Since $\overline{[n]_{q_i}} = n$ and $\overline{\begin{bmatrix} n \\ m \end{bmatrix}_{q_i}} = \binom{n}{m}$ we get the remaining Serre relations.

(2) For $v \in (V_{\mathbf{A}_1})_\mu$ and $h \in P^\vee$, we have

$$(q^h;0)_q v = \frac{q^{\mu(h)}-1}{q-1}v.$$

Taking the classical limit of both sides yields our assertion.

(3) By (2), we have $\bar{h}\bar{v}_\lambda = \lambda(h)\bar{v}_\lambda$ for all $h \in P^\vee$. For each $i \in I$, $\bar{e}_i\bar{v}_\lambda$ is trivially zero. By Proposition 3.3.5 and (1), we get $V^1 = U_1^-\bar{v}_\lambda = U^-\bar{v}_\lambda$, and hence V^1 is a highest weight $U(\mathfrak{g})$-module with highest weight λ and highest weight vector $\bar{v}_\lambda$. □

Summarizing the discussions in Propositions 3.3.7 and 3.4.2, and Theorem 3.4.4, we obtain the following identity between the characters of a $U(\mathfrak{g})$-module and a $U_q(\mathfrak{g})$-module.

Proposition 3.4.5. $\operatorname{ch} V^1 = \operatorname{ch} V^q$.

This shows that the $U_q(\mathfrak{g})$-module V^q can be viewed as a *deformation* of the $U(\mathfrak{g})$-module V^1. The next theorem shows that highest weight $U_q(\mathfrak{g})$-modules in the category $\mathcal{O}_{\text{int}}^q$ and highest weight $U(\mathfrak{g})$-modules in the category $\mathcal{O}_{\text{int}}$ are in good correspondence.

Theorem 3.4.6. *If $\lambda \in P^+$ and V^q is the irreducible highest weight $U_q(\mathfrak{g})$-module $V^q(\lambda)$ with highest weight λ, then V^1 is isomorphic to the irreducible highest weight module $V(\lambda)$ over $U(\mathfrak{g})$ with highest weight λ. Hence, the character of $V^q(\lambda)$ is the same as the character of $V(\lambda)$, which is given by the Weyl-Kac character formula in Theorem* 2.4.6.

Proof. Let v_λ be the highest weight vector of V^q. By Proposition 3.2.6 and Theorem 3.4.4 (3), V^1 is a highest weight $U(\mathfrak{g})$-module with highest weight λ and highest weight vector $\bar{v}_\lambda$ satisfying $f_i^{\lambda(h_i)+1}\bar{v}_\lambda = \bar{f}_i^{\lambda(h_i)+1}\bar{v}_\lambda = 0$ for all $i \in I$. Hence Theorem 2.4.6 shows $V^1 \cong V(\lambda)$. The second assertion follows from Proposition 3.4.5. □

Corollary 3.4.7. *Let $\lambda \in P^+$ and let V^q be a highest weight module over $U_q(\mathfrak{g})$ with highest weight λ and highest weight vector v_λ. If $f_i^{\lambda(h_i)+1}v_\lambda = 0$ for all $i \in I$, then V^q is isomorphic to the irreducible highest weight module $V^q(\lambda)$.*

Proof. As in the proof of Theorem 3.4.6, we have $V^1 \cong V(\lambda)$ as $U(\mathfrak{g})$-modules. Hence, $\operatorname{ch} V^q = \operatorname{ch} V^1 = \operatorname{ch} V(\lambda) = \operatorname{ch} V^q(\lambda)$. Note that there exists a (weight preserving) surjective $U_q(\mathfrak{g})$-module homomorphism $V^q \to V^q(\lambda)$. Since the characters are the same, this must be an isomorphism. □

Corollary 3.4.8.

(1) *If V^q is a highest weight $U_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{\mathrm{int}}$ with highest weight $\lambda \in P$, then $\lambda \in P^+$ and $V^q \cong V^q(\lambda)$.*

(2) *Every irreducible $U_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{\mathrm{int}}$ is isomorphic to $V^q(\lambda)$ for some $\lambda \in P^+$.*

Proof. (1) Under the conditions given, V^1 is a highest weight module in category $\mathcal{O}_{\mathrm{int}}$. Hence by Corollary 2.4.7 we have $V^1 \cong V(\lambda)$ with $\lambda \in P^+$ as $U(\mathfrak{g})$-modules with $\lambda \in P^+$. The rest follows as in the proof for Corollary 3.4.7.

(2) Let V^q be an irreducible $U_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{\mathrm{int}}$. Since $\mathrm{wt}(V^q) \subset \bigcup_{j=1}^{s} D(\lambda_j)$ for some $\lambda_1, \ldots, \lambda_s \in P$, there exists a maximal vector v_λ of weight λ for some $\lambda \in P$. Then v_λ generates a highest weight $U_q(\mathfrak{g})$-module W^q with highest weight λ. By (1), we must have $\lambda \in P^+$ and $W^q \cong V^q(\lambda)$. Since V^q is irreducible, we conclude that $V^q = W^q$. □

Theorem 3.4.9. *The classical limit U_1 of $U_q(\mathfrak{g})$ inherits a Hopf algebra structure from that of $U_q(\mathfrak{g})$, and it is isomorphic to the universal enveloping algebra $U(\mathfrak{g})$ as a Hopf algebra over* **F**.

Proof. By Theorem 3.4.4 (1), there exists a surjective **F**-algebra homomorphism $\psi : U(\mathfrak{g}) \to U_1$ defined by $e_i \mapsto \bar{e}_i$, $f_i \mapsto \bar{f}_i$, and $h \mapsto \bar{h}$ for $i \in I$ and $h \in P^\vee$. Recall from Proposition 2.1.7 that $U(\mathfrak{g}) \cong U^- \otimes U^0 \otimes U^+$.

We first show that the restriction ψ_0 of ψ to U^0 is an isomorphism of U^0 onto U_1^0. The restricted map ψ_0 is certainly surjective. Choose any **Z**-basis $\{x_i\}$ of the free **Z**-lattice $P^\vee$. It is also a basis of the Cartan subalgebra $\mathfrak{h}$.

Thus any element of U^0 may be written as a polynomial in $\{x_i\}$. Suppose $g \in \ker \psi_0$. Then, for each $\lambda \in P$, we have

$$0 = \psi_0(g) \cdot \bar{v}_\lambda = \lambda(g)\bar{v}_\lambda,$$

where v_λ is a highest weight vector of a highest weight $U_q(\mathfrak{g})$-module of highest weight λ and where $\lambda(g)$ denotes the polynomial in $\lambda(x_i)$ corresponding to g. Hence, we have $\lambda(g) = 0$ for every $\lambda \in P$. Since we may take any integer value for $\lambda(x_i)$, g must be zero, which implies that ψ_0 is injective.

Next, we show that the restriction of ψ to U^-, which we denote by ψ_-, is an isomorphism of U^- onto U_1^-. Suppose $\ker \psi_- \neq 0$ and $u = \sum a_\zeta f_\zeta \in \ker \psi_-$, where $a_\zeta \in \mathbf{F}$ and f_ζ are monomials in f_i's ($i \in I$). Let N be the maximal length of the monomials f_ζ in the expression of u, and choose a dominant integral weight $\lambda \in P^+$ such that $\lambda(h_i) > N$ for all $i \in I$. If $V^q = V^q(\lambda)$ is the irreducible $U_q(\mathfrak{g})$-module of highest weight λ, then by Theorem 3.4.6, the representation V^1 is isomorphic to the irreducible $U(\mathfrak{g})$-module $V(\lambda)$ of highest weight λ. By Theorem 2.4.6 and Remark 2.4.5, the kernel of the map $\varphi : U^- \to V^1$, given by $x \mapsto \psi(x) \cdot v_\lambda$, is the left ideal of U^- generated by the elements $f_i^{\lambda(h_i)+1}$ for $i \in I$. Therefore, $u = \sum a_\zeta f_\zeta$ cannot belong to $\ker \varphi$. That is, $\psi_-(u) \cdot v_\lambda = \psi(u) \cdot v_\lambda \neq 0$, which is a contradiction. Therefore, $\ker \psi_- = 0$ and U^- is isomorphic to U_1^-.

Similarly, we have $U^+ \cong U_1^+$. Hence, by the triangular decomposition, we have the linear isomorphisms

$$U(\mathfrak{g}) \cong U^- \otimes U^0 \otimes U^+ \cong U_1^- \otimes U_1^0 \otimes U_1^+ \cong U_1,$$

where the last one follows from Proposition 3.3.3. It is easy to show that this isomorphism is actually an algebra isomorphism (Exercise 3.14).

For the Hopf algebra structure, we first show that U_1 inherits a Hopf algebra structure from that of $U_q(\mathfrak{g})$. It suffices to show that $U_{\mathbf{A}_1}$ inherits the Hopf algebra structure of $U_q(\mathfrak{g})$. This is accomplished by observing that

$$\begin{aligned}
&\Delta((q^h;0)_q) = \frac{q^h \otimes q^h - 1 \otimes 1}{q-1} = (q^h;0)_q \otimes 1 + q^h \otimes (q^h;0)_q, \\
&\varepsilon((q^h;0)_q) = 0, \\
&S((q^h;0)_q) = (q^{-h};0)_q.
\end{aligned} \tag{3.16}$$

Hence the maps $\Delta : U_{\mathbf{A}_1} \to U_{\mathbf{A}_1} \otimes U_{\mathbf{A}_1}$, $\varepsilon : U_{\mathbf{A}_1} \to \mathbf{A}_1$, and $S : U_{\mathbf{A}_1} \to U_{\mathbf{A}_1}$ are all well defined and U_1 inherits a Hopf algebra structure from $U_q(\mathfrak{g})$.

Let us now show that the Hopf algebra structure of $U_q(\mathfrak{g})$ coincides with that of $U(\mathfrak{g})$ under the isomorphism we have been considering. Taking the classical limit of the equations in Proposition 3.1.2 and (3.16), we have

$$\Delta(\bar{h}) = \bar{h} \otimes 1 + 1 \otimes \bar{h},$$
$$\Delta(\bar{e}_i) = \bar{e}_i \otimes 1 + 1 \otimes \bar{e}_i,$$
$$\Delta(\bar{f}_i) = \bar{f}_i \otimes 1 + 1 \otimes \bar{f}_i.$$

This coincides with the comultiplication given in (2.5). The classical limit of other maps may also be checked to coincide with the maps for $U(\mathfrak{g})$. □

Since $U^- \cong U_1^-$, it is natural to expect that the classical limit of a Verma module over $U_q(\mathfrak{g})$ is isomorphic to the Verma module over $U(\mathfrak{g})$ with the same highest weight. This is proved in the next theorem.

Theorem 3.4.10. *Let $\lambda \in P$. If V^q is the Verma module $M^q(\lambda)$ over $U_q(\mathfrak{g})$ with highest weight λ, then its classical limit V^1 is isomorphic to the Verma module $M(\lambda)$ over $U(\mathfrak{g})$ with highest weight λ.*

Proof. Let v_λ be the highest weight vector of V^q. Since $U^- \cong U_1^-$, it suffices to show that V^1 is a free U_1^--module of rank one generated by the highest weight vector $\bar{v}_\lambda$.

Recall from Proposition 3.2.2 that $V^q = M^q(\lambda)$ is a free U_q^--module of rank one generated by the highest weight vector v_λ. Noting the fact that $V_{\mathbf{A}_1}$ is a subspace of V^q and taking Proposition 3.3.5 into account, we see that $V_{\mathbf{A}_1}$ is a free $U_{\mathbf{A}_1}^-$-module generated by v_λ. Taking the classical limit, we see that $V^1 = U_1^- \cdot \bar{v}_\lambda$.

It remains to show that $V^1 \cong V_{\mathbf{A}_1}/\mathbf{J}_1 V_{\mathbf{A}_1}$ is a free U_1^--module. Suppose $\bar{u} \cdot \bar{v}_\lambda = 0$ for some $u \in U_{\mathbf{A}_1}^-$. Then $u \cdot v_\lambda \in \mathbf{J}_1 V_{\mathbf{A}_1} = \mathbf{J}_1 U_{\mathbf{A}_1}^- v_\lambda$. But since $V_{\mathbf{A}_1}$ is a free $U_{\mathbf{A}_1}^-$-module generated by v_λ, we must have $u \in \mathbf{J}_1 U_{\mathbf{A}_1}^-$, which implies $\bar{u} = 0$ in $U_1^- \cong U_{\mathbf{A}_1}^-/\mathbf{J}_1 U_{\mathbf{A}_1}^-$ (see, for example, [**18**, Lemma IV.2.10]). Therefore, V^1 is a free U_1^--module of rank one generated by the highest weight vector $\bar{v}_\lambda$. □

3.5. Complete reducibility of the category $\mathcal{O}^q_{\text{int}}$

In this subsection, we will prove the complete reducibility of $U_q(\mathfrak{g})$-modules in the category $\mathcal{O}^q_{\text{int}}$. We first define the notion of *finite dual* (or *restricted dual*) of a $U_q(\mathfrak{g})$-module. Let V be a $U_q(\mathfrak{g})$-module belonging to the category $\mathcal{O}^q$. It is graded by the weight lattice P with each weight space of finite dimension:

$$V = \bigoplus_{\mu \in P} V_\mu \quad \text{with } \dim V_\mu < \infty. \tag{3.17}$$

We define the ***finite dual*** of V to be the vector space

$$V^* = \bigoplus_{\mu} V_\mu^*, \quad \text{where } V_\mu^* = \mathrm{Hom}_{\mathbf{F}(q)}(V_\mu, \mathbf{F}(q)) \tag{3.18}$$

with the action of $U_q(\mathfrak{g})$ on V^* defined by

$$\langle x \cdot \phi, v\rangle = \langle \phi, S(x)\cdot v\rangle \tag{3.19}$$

for each $x \in U_q(\mathfrak{g})$, $\phi \in V^*$, and $v \in V$. From the property

$$x \cdot V_\mu \subset V_{\mathrm{wt}(x)+\mu},$$

we may show that $x \cdot \phi$ actually belongs to V^*. Although we will not have chances to use the real dual of highest weight modules $V(\lambda)$ or $V^q(\lambda)$, to reduce confusion, we shall write $V^*(\lambda)$ and $V^{q*}(\lambda)$ to denote their finite duals.

Since the antipode S is an anti*auto*morphism, we could have defined the dual space using S^{-1} in place of S. The dual of V thus defined will be denoted by V'.

The following lemma is an immediate consequence of the definitions.

Lemma 3.5.1. *Suppose that V is a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{\mathrm{int}}$ such that $\mathrm{wt}(V) \subset \bigcup_{j=1}^s(\lambda_j - Q_+)$ for some $\lambda_j \in P$ ($j = 1, \dots, s$).*

(1) *There exist canonical isomorphisms $(V^*)' \cong V \cong (V')^*$.*

(2) *The space V_μ^* is a weight space of weight $-\mu$.*

(3) *The finite dual V^* is also integrable and we have*

$$\mathrm{wt}(V^*) \subset \bigcup_{j=1}^{s}(-\lambda_j + Q_+).$$

Proof. We leave it to the readers as an exercise (Exercise 3.16). □

Suppose that V is a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{\mathrm{int}}$. Recalling the definition of category $\mathcal{O}^q_{\mathrm{int}}$, we may choose a *maximal* weight $\lambda \in \mathrm{wt}(V)$ with the property that $\lambda + \alpha_i$ is not a weight for any $i \in I$. Fix any $v_\lambda \in V_\lambda$ and set $L = U_q(\mathfrak{g})v_\lambda$. Then from Corollary 3.4.8 (1) we know $L \cong V^q(\lambda)$ with $\lambda \in P^+$.

Let v_λ^* denote an element in V_λ^* satisfying $v_\lambda^*(v_\lambda) = 1$, $v_\lambda^*(V_\mu) = 0$ for $\lambda \neq \mu$, and set

$$\overline{L} = U_q(\mathfrak{g})v_\lambda^* \subset V^*.$$

Lemma 3.5.2. *The $U_q(\mathfrak{g})$-module $\overline{L}$ is isomorphic to the irreducible lowest weight module $V^{q*}(\lambda)$ with lowest weight $-\lambda$ and lowest weight vector v_λ^*.*

Proof. Lemma 3.5.1 shows that $\overline{L}$ is integrable. Moreover, from the choice of λ, we know v^*_λ is a *lowest* weight vector of weight $-\lambda$. That is, it satisfies

$$f_i v^*_\lambda = 0 \quad \text{for all } i \in I,$$
$$q^h v^*_\lambda = q^{-\lambda(h)} v_\lambda.$$

Hence $\overline{L}$ is an integrable lowest weight module of lowest weight $-\lambda$. Translating the theory of the category $\mathcal{O}^q_{\text{int}}$ to the case of modules with weights bounded below, we know that it is an irreducible lowest weight module of lowest weight $-\lambda$. Since $V^{q*}(\lambda)$ is one such module, the translation of Corollary 3.4.8 (1) tells us that these two modules must be isomorphic. □

We may now single out at least one irreducible component from V.

Lemma 3.5.3. *Let V be a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{\text{int}}$ and let L be the submodule of V generated by a maximal vector v_λ of weight λ. Then we have*

$$V \cong L \oplus V/L.$$

Proof. We will show that in the short exact sequence

$$0 \to L \xrightarrow{\iota} V \to V/L \to 0,$$

the map ι has a left inverse. Let us take the dual with respect to S^{-1} of the injection $\overline{L} \to V^*$ to obtain a map $(V^*)' \to (\overline{L})'$. With the help of Lemma 3.5.1, we may consider the following sequence of maps:

$$L \hookrightarrow V \xrightarrow{\varphi} (\overline{L})'.$$

We may easily check that the image of v_λ is nonzero under the composition of these maps. Using Lemma 3.5.2, we also know that both L and $(\overline{L})'$ are isomorphic to the irreducible highest weight module $V^q(\lambda)$. Hence, by Schur's Lemma, the above composition of maps must be an isomorphism. By composing the inverse of this isomorphism with the map φ, we obtain the left inverse of ι. Hence the above short exact sequence splits and we have

$$V \cong (\overline{L})' \oplus (\ker\varphi) \cong L \oplus V/L.$$

□

We may now use this lemma to show the complete reducibility theorem.

Theorem 3.5.4. *Let $U_q(\mathfrak{g})$ be the quantum group associated with the Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$. Then every $U_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{\text{int}}$ is isomorphic to a direct sum of irreducible highest weight modules $V^q(\lambda)$ with $\lambda \in P^+$.*

Proof. Let $F \subset V$ be a finite dimensional $U_q^{\geq 0}$-submodule and set $V_F = U_q(\mathfrak{g})F \subset V$. We may choose a maximal weight vector of $F \subset V_F$ and apply Lemma 3.5.3 to obtain

$$V_F = L \oplus L_1 \cong L \oplus V_F/L$$

for some irreducible highest weight module L with dominant integral highest weight and its complementary submodule L_1. Note that, as a $U_q(\mathfrak{g})$-module, L_1 is isomorphic to V_F/L which is generated by the $U_q^{\geq 0}$-module $F/(F \cap L)$. Since the dimension of $F/(F \cap L)$ is strictly less than that of F, using induction, we may write the submodule V_F as a direct sum of irreducible highest weight modules with dominant integral highest weights.

Now, for any $v \in V$, by definition of $\mathcal{O}_{\text{int}}^q$, the $U_q^{\geq 0}$-module $F(v) = U_q^{\geq 0}v$ is finite dimensional. Hence, using previous notation, we obtain

$$V = \sum_{v \in V} V_{F(v)},$$

where each $V_{F(v)}$ is a (direct) sum of irreducible highest weight modules with dominant integral highest weights. Thus V can be expressed as a sum of irreducible highest weight modules with dominant integral highest weights. Therefore, by the general argument for semisimplicity ([**8**, Proposition 3.12]), we can deduce that this sum is actually a direct sum, which proves our claim. □

Corollary 3.5.5. *The tensor product of a finite number of $U_q(\mathfrak{g})$-modules in the category $\mathcal{O}_{\text{int}}^q$ is completely reducible.*

Exercises

3.1. Show that the q-integers $[n]_q$ and the q-binomial coefficients $\begin{bmatrix} m \\ n \end{bmatrix}_q$ are elements of $\mathbf{Z}[q, q^{-1}]$ for all nonnegative integers $m \geq n \geq 0$.

3.2. (a) Show that the quantum adjoint operator satisfies

$$(\operatorname{ad}_q e_i)^N(e_j) = \sum_{k=0}^{N} (-1)^k q_i^{k(N+a_{ij}-1)} \begin{bmatrix} N \\ k \end{bmatrix}_{q_i} e_i^{N-k} e_j e_i^k.$$

(b) Verify that the algebra homomorphism Δ defined in Proposition 3.1.2 satisfies

$$\begin{aligned}\Delta((\mathrm{ad}_q\, e_i)^N(e_j)) &= (\mathrm{ad}_q\, e_i)^N(e_j)\otimes K_i^{-N}K_j^{-1}\\ &+\sum_{k=0}^{N-1}\tau_k^{(N)} q_i^{k(N-k)}\begin{bmatrix} N\\ k\end{bmatrix}_{q_i} e_i^{N-k}\otimes K_i^{-N+k}(\mathrm{ad}_q\, e_i)^k(e_j)\\ &+1\otimes(\mathrm{ad}_q\, e_i)^N(e_j),\end{aligned}$$

where $\tau_k^{(N)}=\prod_{t=k}^{N-1}(1-q_i^{2(t+a_{ij})})$.

3.3. Verify that the maps Δ, ε, and S defined in Proposition 3.1.2 satisfy all the conditions for Hopf algebras.

3.4. Prove $U_q^{\geq 0}\cong U_q^0\otimes U_q^+$.

3.5. Show that, for each $i\in I$, every $U_q(\mathfrak{g})$-module V^q in the category $\mathcal{O}_{\mathrm{int}}^q$ decomposes into a direct sum of finite dimensional irreducible $U_q(\mathfrak{g}_{(i)})$-submodules, where $U_q(\mathfrak{g}_{(i)})$ is the subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i, $K_i^{\pm 1}$.

3.6. Prove the following commutation relation for $k,l\in\mathbf{Z}_{\geq 0}$:

$$e_i^{(k)}f_i^{(l)}=\sum_{t=0}^{\min(k,l)}\frac{1}{[t]_{q_i}!}f_i^{(l-t)}\left(\prod_{s=1}^{t}[K_i;(t+s)-(k+l)]_{q_i}\right)e_i^{(k-t)}.$$

3.7. For $u\in(U_q^-)_{-\alpha}$ with $\alpha\in Q_+$, verify that

$$f_i^n u=\sum_{k=0}^{n} q_i^{(\alpha(h_i)+k)(n-k)}\begin{bmatrix} n\\ k\end{bmatrix}_{q_i}\left((\mathrm{ad}_q\, f_i)^k(u)\right)f_i^{n-k}.$$

3.8. Show that the identities in the proof of Lemma 3.3.2 hold.

3.9. Verify the three commutation relations stated in the proof of Proposition 3.3.3.

3.10. (a) Show that $\mathbf{A}_1$ is a principal ideal domain. Deduce that, for each $\mu\in P$, the weight space $(V_{\mathbf{A}_1})_\mu$ is a free $\mathbf{A}_1$-module.

(b) Let $\mathbf{A}$ be an integral domain and let $\mathbf{F}$ be its field of quotients. Consider a vector space V over $\mathbf{F}$. Show that a set of vectors $\{v_1,\dots,v_n\}$ is $\mathbf{F}$-linearly independent if and only if it is $\mathbf{A}$-linearly independent.

(c) Deduce that $\mathrm{rank}_{\mathbf{A}_1}(V_{\mathbf{A}_1})_\mu=\dim_{\mathbf{F}(q)}V_\mu^q$.

3.11. Let $\mathbf{A}$ be a commutative ring with 1 and V be a free $\mathbf{A}$-module with basis $\{v_j\,|\,j\in J\}$. Show that, for any $\mathbf{A}$-module W, every element u of $W\otimes V$ can be expressed uniquely as

$$u=\sum_{j\in J}w_j\otimes v_j\quad\text{with } w_j\in W.$$

3.12. (a) Show that for each $\mu \in P$, a set $\{v_i\} \subset (V_{\mathbf{A}_1})_\mu$ is a basis of V_μ^q over $\mathbf{F}(q)$ if the set $\{\bar{v}_i\}$ is a basis of V_μ^1 over $\mathbf{F}$.

(b) Verify that the map $V^1 \to V_{\mathbf{A}_1}$ given by $(c(q)+\mathbf{J}_1)\otimes v \mapsto c(1)v$ is injective.

(c) View V^1 as being embedded in $V_{\mathbf{A}_1}$ and hence in V^q. For each $\mu \in P$, any basis of V_μ^1 over $\mathbf{F}$ is a basis of V_μ^q over $\mathbf{F}(q)$.

3.13. As with the modules, we may view $U^\pm \cong U_1^\pm$ as a subset of $U_q^\pm$. Define $(U_{\mathbf{A}_1}^\pm)_{\pm\mu} = U_{\mathbf{A}_1}^\pm \cap (U_q^\pm)_{\pm\mu}$ and $(U_1^\pm)_{\pm\mu} = \mathbf{F} \otimes_{\mathbf{A}_1} (U_{\mathbf{A}_1}^\pm)_{\pm\mu}$ for each $\mu \in Q_+$.

(a) For each $\mu \in Q_+$, show that a set $\{x_i\} \subset (U_{\mathbf{A}_1}^\pm)_{\pm\mu}$ is a basis of $(U_q^\pm)_{\pm\mu}$ if the set $\{\bar{x}_i\}$ is a basis of $(U_1^\pm)_\mu$.

(b) Deduce that, for each $\mu \in Q_+$, any basis of $U_{\pm\mu}^\pm$ is a basis of $(U_q^\pm)_{\pm\mu}$.

3.14. Show that the composition of linear isomorphisms

$$U(\mathfrak{g}) \cong U^- \otimes U^0 \otimes U^+ \cong U_1^- \otimes U_1^0 \otimes U_1^+ \cong U_1$$

is an isomorphism of algebras.

3.15. Let V^q be a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}_{\text{int}}^q$. Show that $\dim V_{w\lambda}^q = \dim V_\lambda^q$ for all $w \in W$, $\lambda \in \text{wt}(V^q)$.

3.16. Prove the statements in Lemma 3.5.1.