Chapter 4
Algebras and Representations

Abstract In this chapter we develop some algebraic tools needed for the general theory of representations and invariants. The central result is a duality theorem for locally regular representations of a reductive algebraic group G. The duality between the irreducible regular representations of G and irreducible representations of the commuting algebra of G plays a fundamental role in classical invariant theory. We study the representations of a finite group through its group algebra and characters, and we construct induced representations and calculate their characters.

4.1 Representations of Associative Algebras

In this section we obtain the basic facts about representations of associative algebras: a general version of Schur's lemma, the Jacobson density theorem, the notion of complete reducibility of representations, the double commutant theorem, and the isotypic decomposition of a locally completely reducible representation of an algebraic group.

4.1.1 Definitions and Examples

We know from the previous chapter that every regular representation (ρ, V) of a reductive linear algebraic group G decomposes into a direct sum of irreducible representations (in particular, this is true when G is a classical group). The same is true for finite-dimensional representations of a semisimple Lie algebra $\mathfrak{g}$. The next task is to determine the extent of uniqueness of such a decomposition and to find explicit projection operators onto irreducible subspaces of V. In the tradition of modern mathematics we will attack these problems by putting them in a more general (abstract) context, which we have already employed, for example, in the proof of the theorem of the highest weight in Section 3.2.1.

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Definition 4.1.1. An *associative algebra* over the complex field $\mathbb{C}$ is a vector space $\mathcal{A}$ over $\mathbb{C}$ together with a bilinear multiplication map

$$\mu : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}\,, \quad x, y \mapsto xy = \mu(x,y)\,,$$

such that $(xy)z = x(yz)$. The algebra $\mathcal{A}$ is said to have a *unit element* if there exists $e \in \mathcal{A}$ such that $ae = ea = a$ for all $a \in \mathcal{A}$. If $\mathcal{A}$ has a unit element it is unique and it will usually be denoted by 1.

Examples

1. Let V be a vector space over $\mathbb{C}$ (possibly infinite-dimensional), and let $\mathcal{A} = \mathrm{End}(V)$ be the space of $\mathbb{C}$-linear transformations on V. Then $\mathcal{A}$ is an associative algebra with multiplication the composition of transformations. When $\dim V = n < \infty$, then this algebra has a basis consisting of the *elementary matrices* e_{ij} that multiply by $e_{ij}e_{km} = \delta_{jk}e_{im}$ for $1 \le i, j \le n$. This algebra will play a fundamental role in our study of associative algebras and their representations.

2. Let G be a group. We define an associative algebra $\mathcal{A}[G]$, called the *group algebra* of G, as follows: As a vector space, $\mathcal{A}[G]$ is the set of all functions $f : G \longrightarrow \mathbb{C}$ such that the *support* of f (the set where $f(g) \neq 0$) is finite. This space has a basis consisting of the functions $\{\delta_g : g \in G\}$, where

$$\delta_g(x) = \begin{cases} 1 & \text{if } x = g\,, \\ 0 & \text{otherwise.} \end{cases}$$

Thus an element x of $\mathcal{A}[G]$ has a unique expression as a formal sum $\sum_{g\in G} x(g)\,\delta_g$ with only a finite number of coefficients $x(g) \neq 0$.

We identify $g \in G$ with the element $\delta_g \in \mathcal{A}[G]$, and we define multiplication on $\mathcal{A}[G]$ as the bilinear extension of group multiplication. Thus, given functions $x, y \in \mathcal{A}[G]$, we define their product $x * y$ by

$$\big(\textstyle\sum_{g\in G} x(g)\,\delta_g\big) * \big(\textstyle\sum_{h\in G} y(h)\,\delta_h\big) = \textstyle\sum_{g,h\in G} x(g)y(h)\,\delta_{gh}\,,$$

with the sum over $g, h \in G$. (We indicate the multiplication by $*$ so it will not be confused with the pointwise multiplication of functions on G.) This product is associative by the associativity of group multiplication. The identity element $e \in G$ becomes the unit element δ_e in $\mathcal{A}[G]$ and G is a subgroup of the group of invertible elements of $\mathcal{A}[G]$. The function $x * y$ is called the *convolution* of the functions x and y; from the definition it is clear that

$$(x * y)(g) = \textstyle\sum_{hk=g} x(h)y(k) = \sum_{h\in G} x(h)y(h^{-1}g)\,.$$

If $\varphi : G \longrightarrow H$ is a group homomorphism, then we can extend φ uniquely to a linear map $\widetilde{\varphi} : \mathcal{A}[G] \longrightarrow \mathcal{A}[H]$ by the rule

$$\widetilde{\varphi}\left(\sum_{g\in G} x(g)\delta_g\right) = \sum_{g\in G} x(g)\delta_{\varphi(g)}\,.$$

From the definition of multiplication in $\mathcal{A}[G]$ we see that the extended map $\widetilde{\varphi}$ is an associative algebra homomorphism. Furthermore, if $\psi : H \longrightarrow K$ is another group homomorphism, then $\widetilde{\psi \circ \varphi} = \widetilde{\psi} \circ \widetilde{\varphi}$.

An important special case occurs when G is a subgroup of H and φ is the inclusion map. Then $\widetilde{\varphi}$ is injective (since $\{\delta_g\}$ is a basis of $\mathcal{A}[G]$). Thus we can identify $\mathcal{A}[G]$ with the subalgebra of $\mathcal{A}[H]$ consisting of functions supported on G.

3. Let $\mathfrak{g}$ be a Lie algebra over $\mathcal{A}$. Just as in the case of group algebras, there is an associative algebra $U(\mathfrak{g})$ (the *universal enveloping algebra* of $\mathfrak{g}$) and an injective linear map $j : \mathfrak{g} \longrightarrow U(\mathfrak{g})$ such that $j(\mathfrak{g})$ generates $U(\mathfrak{g})$ and

$$j([X,Y]) = j(X)j(Y) - j(Y)j(X)$$

(the multiplication on the right is in $U(\mathfrak{g})$; see Appendix C.2.1 and Theorem C.2.2). Since $U(\mathfrak{g})$ is uniquely determined by $\mathfrak{g}$, up to isomorphism, we will identify $\mathfrak{g}$ with $j(\mathfrak{g})$. If $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra then the Poincaré–Birkhoff–Witt Theorem C.2.2 allows us to identify $U(\mathfrak{h})$ with the associative subalgebra of $U(\mathfrak{g})$ generated by $\mathfrak{h}$, so we have the same situation as for the group algebra of a subgroup $H \subset G$.

Definition 4.1.2. Let $\mathcal{A}$ be an associative algebra over $\mathbb{C}$. A *representation* of $\mathcal{A}$ is a pair (ρ, V), where V is a vector space over $\mathbb{C}$ and $\rho : \mathcal{A} \longrightarrow \mathrm{End}(V)$ is an associative algebra homomorphism. If $\mathcal{A}$ has an identity element 1, then we require that $\rho(1)$ act as the identity transformation I_V on V.

When the map ρ is understood from the context, we shall call V an *$\mathcal{A}$-module* and write av for $\rho(a)v$. If V, W are both $\mathcal{A}$-modules, then we make the vector space $V \oplus W$ into an $\mathcal{A}$-module by the action $a \cdot (v \oplus w) = av \oplus aw$.

If $U \subset V$ is a linear subspace such that $\rho(a)U \subset U$ for all $a \in \mathcal{A}$, then we say that U is *invariant* under the representation. In this case we can define a representation (ρ_U, U) by the restriction of $\rho(\mathcal{A})$ to U and a representation $(\rho_{V/U}, V/U)$ by the natural quotient action of $\rho(\mathcal{A})$ on V/U. A representation (ρ, V) is *irreducible* if the only invariant subspaces are $\{0\}$ and V.

Define $\mathrm{Ker}(\rho) = \{x \in \mathcal{A} : \rho(x) = 0\}$. This is a two-sided ideal in $\mathcal{A}$, and V is a module for the quotient algebra $\mathcal{A}/\mathrm{Ker}(\rho)$ via the natural quotient map. A representation ρ is *faithful* if $\mathrm{Ker}(\rho) = 0$.

Definition 4.1.3. Let (ρ, V) and (τ, W) be representations of $\mathcal{A}$, and let $\mathrm{Hom}(V,W)$ be the space of $\mathbb{C}$-linear maps from V to W. We denote by $\mathrm{Hom}_{\mathcal{A}}(V,W)$ the set of all $T \in \mathrm{Hom}(V,W)$ such that $T\rho(a) = \tau(a)T$ for all $a \in \mathcal{A}$. Such a map is called an *intertwining operator* between the two representations or a *module homomorphism*.

If $U \subset V$ is an invariant subspace, then the inclusion map $U \longrightarrow V$ and the quotient map $V \longrightarrow V/U$ are intertwining operators. We say that the representations (ρ, V) and (τ, W) are *equivalent* if there exists an invertible operator in $\mathrm{Hom}_{\mathcal{A}}(V,W)$. In this case we write $(\rho, V) \cong (\tau, W)$.

The composition of two intertwining operators, when defined, is again an intertwining operator. In particular, when $V = W$ and $\rho = \tau$, then $\mathrm{Hom}_{\mathcal{A}}(V,V)$ is an associative algebra, which we denote by $\mathrm{End}_{\mathcal{A}}(V)$.

Examples

1. Let $\mathcal{A} = \mathbb{C}[x]$ be the polynomial ring in one indeterminate. Let V be a finite-dimensional vector space, and let $T \in \mathrm{End}(V)$. Define a representation (ρ, V) of $\mathcal{A}$ by $\rho(f) = f(T)$ for $f \in \mathbb{C}[x]$. Then $\mathrm{Ker}(\rho)$ is the ideal in $\mathcal{A}$ generated by the *minimal polynomial* of T. The problem of finding a canonical form for this representation is the same as finding the Jordan canonical form for T (see Section B.1.2).

2. Let G be a group and let $\mathcal{A} = \mathcal{A}[G]$ be the group algebra of G. If (ρ, V) is a representation of $\mathcal{A}$, then the map $g \mapsto \rho(\delta_g)$ is a group homomorphism from G to $\mathbf{GL}(V)$. Conversely, every representation $\rho : G \longrightarrow \mathbf{GL}(V)$ extends uniquely to a representation ρ of $\mathcal{A}[G]$ on V by

$$\rho(f) = \sum_{g \in G} f(g)\rho(g)$$

for $f \in \mathcal{A}[G]$. We shall use the same symbol to denote a representation of a group and its group algebra.

Suppose $W \subset V$ is a linear subspace. If W is invariant under G and $w \in W$, then $\rho(f)w \in W$, since $\rho(g)w \in W$. Conversely, if $\rho(f)W \subset W$ for all $f \in \mathcal{A}[G]$, then $\rho(G)W \subset W$, since we can take $f = \delta_g$ with g arbitrary in G. Furthermore, an operator $R \in \mathrm{End}(V)$ commutes with the action of G if and only if it commutes with $\rho(f)$ for all $f \in \mathcal{A}[G]$.

Two important new constructions are possible in the case of group representations (we already encountered them in Section 1.5.1 when G is a linear algebraic group). The first is the *contragredient* or *dual* representation (ρ^*, V^*), where

$$\langle \rho^*(g)f, v\rangle = \langle f, \rho(g^{-1})v\rangle$$

for $g \in G$, $v \in V$, and $f \in V^*$. The second is the *tensor product* $(\rho \otimes \sigma, V \otimes W)$ of two representations defined by

$$(\rho \otimes \sigma)(g)(v \otimes w) = \rho(g)v \otimes \sigma(g)w\,.$$

For example, let (ρ, V) and (σ, W) be finite-dimensional representations of G. There is a representation π of G on $\mathrm{Hom}(V, W)$ by $\pi(g)T = \sigma(g)T\rho(g)^{-1}$ for $T \in \mathrm{Hom}(V, W)$. There is a natural linear isomorphism

$$\mathrm{Hom}(V, W) \cong W \otimes V^* \tag{4.1}$$

(see Section B.2.2). Here a tensor of the form $w \otimes v^*$ gives the linear transformation $Tv = \langle v^*, v\rangle w$ from V to W. Since the tensor $\sigma(g)w \otimes \rho^*(g)v^*$ gives the linear transformation

$$v \mapsto \langle \rho^*(g)v^*, v\rangle \sigma(g)w = \langle v^*, \rho(g)^{-1}v\rangle \sigma(g)w = \sigma(g)T\rho(g)^{-1}v\,,$$

we see that π is equivalent to $\sigma \otimes \rho^*$. In particular, the space $\mathrm{Hom}_G(V,W)$ of G-intertwining maps between V and W corresponds to the space $(W \otimes V^*)^G$ of G-fixed elements in $W \otimes V^*$.

We can iterate the tensor product construction to obtain G-modules $\bigotimes^k V = V^{\otimes k}$ (the k-fold tensor product of V with itself) with $g \in G$ acting by

$$\rho^{\otimes k}(g)(v_1 \otimes \cdots \otimes v_k) = \rho(g)v_1 \otimes \cdots \otimes \rho(g)v_k$$

on decomposable tensors. The subspaces $S^k(V)$ (symmetric tensors) and $\bigwedge^k V$ (skew-symmetric tensors) are G-invariant (see Sections B.2.3 and B.2.4). These modules are called the *symmetric* and *skew-symmetric* powers of ρ.

The contragredient and tensor product constructions for group representations are associated with the *inversion map* $g \mapsto g^{-1}$ and the *diagonal map* $g \mapsto (g,g)$. The properties of these maps can be described axiomatically using the notion of a *Hopf algebra* (see Exercises 4.1.8).

3. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$, and let (ρ, V) be a representation of $\mathfrak{g}$. The universal mapping property implies that ρ extends uniquely to a representation of $U(\mathfrak{g})$ (see Section C.2.1) and that every representation of $\mathfrak{g}$ comes from a unique representation of $U(\mathfrak{g})$, just as in the case of group algebras. In this case we define the *dual* representation (ρ^*, V^*) by

$$\langle \rho^*(X)f, v\rangle = -\langle f, \rho(X)v\rangle \quad \text{for } X \in \mathfrak{g} \text{ and } f \in V^*\,.$$

We can also define the *tensor product* $(\rho \otimes \sigma,\ V \otimes W)$ of two representations by letting $X \in \mathfrak{g}$ act by

$$X \cdot (v \otimes w) = \rho(X)v \otimes w + v \otimes \sigma(X)w\,.$$

When $\mathfrak{g}$ is the Lie algebra of a linear algebraic group G and ρ, σ are the differentials of regular representations of G, then this action of $\mathfrak{g}$ is the differential of the tensor product of the G representations (see Sections 1.5.2).

These constructions are associated with the maps $X \mapsto -X$ and $X \mapsto X \otimes I + I \otimes X$. As in the case of group algebras, the properties of these maps can be described axiomatically using the notion of a *Hopf algebra* (see Exercises 4.1.8). The k-fold tensor powers of ρ and the symmetric and skew-symmetric powers are defined by analogy with the case of group representations. Here $X \in \mathfrak{g}$ acts by

$$\begin{aligned}\rho^{\otimes k}(X)(v_1 \otimes \cdots \otimes v_k) &= \rho(X)v_1 \otimes \cdots \otimes v_k + v_1 \otimes \rho(X)v_2 \otimes \cdots \otimes v_k \\ &\quad + \cdots + v_1 \otimes \cdots \otimes \rho(X)v_k\end{aligned}$$

on decomposable tensors. This action extends linearly to all tensors.

4.1.2 Schur's Lemma

We say that a vector space has *countable dimension* if the cardinality of every linear independent set of vectors is countable.

Lemma 4.1.4. *Let (ρ, V) and (τ, W) be irreducible representations of an associative algebra $\mathcal{A}$. Assume that V and W have countable dimension over $\mathbb{C}$. Then*

$$\dim \mathrm{Hom}_{\mathcal{A}}(V,W) = \begin{cases} 1 \text{ if } (\rho,V) \cong (\tau,W), \\ 0 \text{ otherwise.} \end{cases}$$

Proof. Let $T \in \mathrm{Hom}_{\mathcal{A}}(V,W)$. Then $\mathrm{Ker}(T)$ and $\mathrm{Range}(T)$ are invariant subspaces of V and W, respectively. If $T \neq 0$, then $\mathrm{Ker}(T) \neq V$ and $\mathrm{Range}(T) \neq 0$. Hence by the irreducibility of the representations, $\mathrm{Ker}(T) = 0$ and $\mathrm{Range}(T) = W$, so that T is a linear isomorphism. Thus $\mathrm{Hom}_{\mathcal{A}}(V,W) \neq 0$ if and only if $(\rho,V) \cong (\tau,W)$.

Suppose the representations are equivalent. If $S, T \in \mathrm{Hom}_{\mathcal{A}}(V,W)$ are nonzero, then $R = T^{-1}S \in \mathrm{End}_{\mathcal{A}}(V)$. Assume, for the sake of contradiction, that R is not a multiple of the identity operator. Then for all $\lambda \in \mathbb{C}$ we would have $R - \lambda I$ nonzero and hence invertible. We assert that this implies that for any nonzero vector $v \in V$ and distinct scalars $\lambda_1, \ldots, \lambda_m$, the set

$$\{(R-\lambda_1 I)^{-1}v, \ldots, (R-\lambda_m I)^{-1}v\} \tag{4.2}$$

is linearly independent. We note that this would contradict the countable dimensionality of V and the lemma would follow.

Thus it suffices to prove the linear independence of (4.2) under the hypothesis on R. Suppose there is a linear relation

$$\sum_{i=1}^{m} a_i (R-\lambda_i I)^{-1} v = 0\,.$$

Multiplying through by $\prod_j (R-\lambda_j I)$, we obtain the relation $f(R)v = 0$, where

$$f(x) = \sum_{i=1}^{m} a_i \Big\{ \prod_{j \neq i} (x-\lambda_j) \Big\}\,.$$

The polynomial $f(x)$ takes the value $a_i \prod_{j\neq i}(\lambda_i - \lambda_j)$ at $x = \lambda_i$. If $a_i \neq 0$ for some i, then $f(x)$ is a nonzero polynomial and has a factorization

$$f(x) = c(x-\mu_1)\cdots(x-\mu_m)\,,$$

with $c \neq 0$ and $\mu_i \in \mathbb{C}$. But by our assumption on R the operators $R - \mu_i I$ are invertible for each i, and hence $f(R)$ is invertible. This contradicts the relation $f(R)v = 0$. Thus $a_i = 0$ for all i and the set (4.2) is linearly independent. □

4.1.3 Jacobson Density Theorem

If V is a complex vector space, $v_j \in V$, and $T \in \mathrm{End}(V)$, then we write

$$V^{(n)} = \underbrace{V \oplus \cdots \oplus V}_{n \text{ copies}} \quad \text{and} \quad T^{(n)}[v_1, \ldots, v_n] = [Tv_1, \ldots, Tv_n] .$$

The map $T \mapsto T^{(n)}$ is a representation of $\mathrm{End}(V)$ on $V^{(n)}$. If Z is a subspace of V, then we identify $Z^{(n)}$ with the subspace $\{[z_1, \ldots, z_n] : z_j \in Z\}$ of $V^{(n)}$. If $\mathcal{R} \subset \mathrm{End}(V)$ is a subalgebra, then we consider $V^{(n)}$ to be an $\mathcal{R}$-module with $r \in \mathcal{R}$ acting as $r^{(n)}$; we write rv for $r^{(n)}v$ when $v \in V^{(n)}$.

Theorem 4.1.5. *Let V be a countable-dimensional vector space over $\mathbb{C}$. Let $\mathcal{R}$ be a subalgebra of* $\mathrm{End}(V)$ *that acts irreducibly on V. Assume that for every finite-dimensional subspace W of V there exists $r \in \mathcal{R}$ so that $r|_W = I|_W$. Then $\mathcal{R}[v_1, \ldots, v_n] = V^{(n)}$ whenever $\{v_1, \ldots, v_n\}$ is a linearly independent subset of V.*

Proof. The proof is by induction on n. If $n = 1$ the assertion is the definition of irreducibility. Assume that the theorem holds for n and suppose $\{v_1, \ldots, v_{n+1}\}$ is a linearly independent set in V. Given any elements $x_1, \ldots, x_{n+1}$ in V, we must find $r \in \mathcal{R}$ such that

$$rv_j = x_j \quad \text{for } j = 1, \ldots, n+1 . \tag{4.3}$$

The inductive hypothesis implies that there is an element $r_0 \in \mathcal{R}$ such that $r_0 v_j = x_j$ for $j = 1, \ldots, n$. Define $\mathcal{B} = \{r \in \mathcal{R} : r[v_1, \ldots, v_n] = 0\}$. The subspace $\mathcal{B}v_{n+1}$ of V is invariant under $\mathcal{R}$. Suppose $\mathcal{B}v_{n+1} \neq 0$; then $\mathcal{B}v_{n+1} = V$, since $\mathcal{R}$ acts irreducibly on V. Hence there exists $b_0 \in \mathcal{B}$ such that $b_0 v_{n+1} = x_{n+1} - r_0 v_{n+1}$. Since $b_0 v_j = 0$ for $j = 1, \ldots, n$, we see that the element $r = r_0 + b_0$ of $\mathcal{R}$ satisfies (4.3), and we are done in this case.

To complete the inductive step, it thus suffices to show that $\mathcal{B}v_{n+1} \neq 0$. We assume the contrary and show that this leads to a contradiction. Set

$$W = \mathcal{R}[v_1, \ldots, v_n, v_{n+1}] \quad \text{and} \quad U = \{[\underbrace{0, \ldots, 0}_{n}, v] : v \in V\} .$$

Then $[v_1, \ldots, v_n, v_{n+1}] \in W$. By the inductive hypothesis $V^{(n+1)} = W + U$. If $r \in \mathcal{R}$ and $[rv_1, \ldots, rv_n, rv_{n+1}] \in U \cap W$, then $rv_j = 0$ for $j = 1, \ldots, n$. Hence $r \in \mathcal{B}$ and consequently $rv_{n+1} = 0$ by the assumption $\mathcal{B}v_{n+1} = 0$. Thus $W \cap U = 0$, so we conclude that

$$V^{(n+1)} \cong W \oplus U \tag{4.4}$$

as an $\mathcal{R}$ module. Let $P : V^{(n+1)} \longrightarrow W$ be the projection corresponding to this direct sum decomposition. Then P commutes with the action of $\mathcal{R}$ and can be written as

$$P[x_1, \ldots, x_{n+1}] = \left[\textstyle\sum_j P_{1,j} x_j, \ldots, \sum_j P_{n+1,j} x_j\right]$$

with $P_{i,j} \in \mathrm{End}_{\mathcal{R}}(V)$. Thus by Lemma 4.1.4, each operator $P_{i,j}$ equals $q_{i,j}I$ for some scalar $q_{i,j} \in \mathbb{C}$. Hence for any subspace Z of V we have $P(Z^{(n+1)}) \subset Z^{(n+1)}$.

We can now obtain the desired contradiction. Set $Z = \mathrm{Span}\{v_1, \ldots, v_{n+1}\}$ and let $w_1, \ldots, w_{n+1}$ be arbitrary elements of V. Since $\{v_1, \ldots, v_{n+1}\}$ is linearly independent, there is a linear transformation $T : Z \longrightarrow V$ with $Tv_j = w_j$ for $j = 1, \ldots, n+1$. We calculate

$$\begin{aligned} T^{(n+1)}P[v_1,\ldots,v_{n+1}] &= \left[\sum_j q_{1,j} T v_j, \ldots, \sum_j q_{n+1,j} v_j\right] \\ &= \left[\sum_j q_{1,j} w_j, \ldots, \sum_j q_{n+1,j} w_j\right] \\ &= P[w_1,\ldots,w_{n+1}]\,. \end{aligned}$$

On the other hand,

$$P[v_1,\ldots,v_{n+1}] = [v_1,\ldots,v_{n+1}] \quad \text{and} \quad T^{(n+1)}[v_1,\ldots,v_{n+1}] = [w_1,\ldots,w_{n+1}]\,,$$

so we conclude that $[w_1,\ldots,w_{n+1}] = P[w_1,\ldots,w_{n+1}]$. Hence $[w_1,\ldots,w_{n+1}] \in W$. Since w_j are any elements of V, this implies that $W = V^{(n+1)}$, which contradicts (4.4). □

Corollary 4.1.6. *If X is a finite-dimensional subspace of V and $f \in \mathrm{Hom}(X,L)$, then there exists $r \in \mathcal{R}$ such that $f = r|_X$.*

Proof. Let $\{v_1, \ldots, v_n\}$ be a basis for X and set $w_j = f(v_j)$ for $j = 1, \ldots, n$. By Theorem 4.1.5 there exists $r \in \mathcal{R}$ such that $rv_j = w_j$ for $j = 1, \ldots, n$. Hence by linearity $r|_X = f$. □

Corollary 4.1.7 (Burnside's Theorem). *If $\mathcal{R}$ acts irreducibly on L and* $\dim L < \infty$, *then* $\mathcal{R} = \mathrm{End}(L)$.

Thus the image of an associative algebra in a finite-dimensional irreducible representation (ρ, L) is completely determined by $\dim L$ (the *degree* of the representation).

4.1.4 Complete Reducibility

Let (ρ, V) be a finite-dimensional representation of the associative algebra $\mathcal{A}$. When $V = W \oplus U$ with W and U invariant subspaces, then $U \cong V/W$ as an $\mathcal{A}$-module. In general, if $W \subset V$ is an $\mathcal{A}$-invariant subspace, then by extending a basis for W to a basis for V, we obtain a *vector-space isomorphism* $V \cong W \oplus (V/W)$. However, this isomorphism is not necessarily an isomorphism of $\mathcal{A}$-modules.

Definition 4.1.8. A finite-dimensional $\mathcal{A}$-module V is *completely reducible* if for every $\mathcal{A}$-invariant subspace $W \subset V$ there exists a complementary invariant subspace $U \subset V$ such that $V = W \oplus U$.

We proved in Chapter 3 that rational representations of classical groups and finite-dimensional representations of semisimple Lie algebras are completely reducible. For any associative algebra the property of complete reducibility is inherited by subrepresentations and quotient representations.

Lemma 4.1.9. *Let (ρ, V) be completely reducible and suppose $W \subset V$ is an invariant subspace. Set $\sigma(x) = \rho(x)|_W$ and $\pi(x)(v+W) = \rho(x)v + W$ for $x \in A$ and $v \in V$. Then the representations (σ, W) and $(\pi, V/W)$ are completely reducible.*

Proof. The proof of Lemma 3.3.2 applies verbatim to this context. □

Remark 4.1.10. The converse to Lemma 4.1.9 is not true. For example, let A be the algebra of matrices of the form $\left[\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right]$ with $x, y \in \mathbb{C}$, acting on $V = \mathbb{C}^2$ by left multiplication. The space $W = \mathbb{C}e_1$ is invariant and irreducible. Since V/W is one-dimensional, it is also irreducible. But the matrices in A have only one distinct eigenvalue and are not diagonal, so there is no invariant complement to W in V. Thus V is not completely reducible as an A-module.

Proposition 4.1.11. *Let (ρ, V) be a finite-dimensional representation of the associative algebra A. The following are equivalent:*

1. *(ρ, V) is completely reducible.*
2. *$V = W_1 \oplus \cdots \oplus W_s$ with each W_i an irreducible A-module.*
3. *$V = V_1 + \cdots + V_d$ as a vector space, where each V_i is an irreducible A-submodule.*

Furthermore, if V satisfies these conditions and if all the V_i in (3) are equivalent to a single irreducible A-module W, then every A-submodule of V is isomorphic to a direct sum of copies of W.

Proof. The equivalence of the three conditions follows by the proof of Proposition 3.3.3. Now assume that V satisfies these conditions and that the V_i are all mutually equivalent as A-modules. Let M be an A-submodule of V. Since V is completely reducible by (1), it follows from Lemma 4.1.9 that M is completely reducible. Hence by (2) we have $M = W_1 \oplus \cdots \oplus W_r$ with W_i an irreducible A-module. Furthermore, there is a complementary A-submodule N such that $V = M \oplus N$. Hence

$$V = W_1 \oplus \cdots \oplus W_r \oplus N .$$

Let $p_i : V \longrightarrow W_i$ be the projection corresponding to this decomposition. By (3) we have $W_i = p_i(V_1) + \cdots + p_i(V_d)$. Thus for each i there exists j such that $p_i(V_j) \neq (0)$. Since W_i and V_j are irreducible and p_i is an A-module map, Schur's lemma implies that $W_i \cong V_j$ as an A-module. Hence $W_i \cong W$ for all i. □

Corollary 4.1.12. *Suppose (ρ, V) and (σ, W) are completely reducible representations of A. Then $(\rho \oplus \sigma, V \oplus W)$ is a completely reducible representation.*

Proof. This follows from the equivalence between conditions (1) and (2) in Proposition 4.1.11. □

4.1.5 Double Commutant Theorem

Let V be a vector space. For any subset $\mathcal{S} \subset \mathrm{End}(V)$ we define

$$\mathrm{Comm}(\mathcal{S}) = \{x \in \mathrm{End}(V) : xs = sx \quad \text{for all } s \in \mathcal{S}\}$$

and call it the *commutant* of $\mathcal{S}$. We observe that $\mathrm{Comm}(\mathcal{S})$ is an associative algebra with unit I_V.

Theorem 4.1.13 (Double Commutant). *Suppose* $\mathcal{A} \subset \mathrm{End}V$ *is an associative algebra with identity* I_V. *Set* $\mathcal{B} = \mathrm{Comm}(\mathcal{A})$. *If* V *is a completely reducible* $\mathcal{A}$*-module, then* $\mathrm{Comm}(\mathcal{B}) = \mathcal{A}$.

Proof. By definition we have $\mathcal{A} \subset \mathrm{Comm}(\mathcal{B})$. Let $T \in \mathrm{Comm}(\mathcal{B})$ and fix a basis $\{v_1, \ldots, v_n\}$ for V. It will suffice to find an element $S \in \mathcal{A}$ such that $Sv_i = Tv_i$ for $i = 1, \ldots, n$. Let $w_0 = v_1 \oplus \cdots \oplus v_n \in V^{(n)}$. Since $V^{(n)}$ is a completely reducible $\mathcal{A}$-module by Proposition 4.1.11, the cyclic submodule $M = \mathcal{A} \cdot w_0$ has an $\mathcal{A}$-invariant complement. Thus there is a projection $P : V^{(n)} \longrightarrow M$ that commutes with $\mathcal{A}$. The action of P is given by an $n \times n$ matrix $[p_{ij}]$, where $p_{ij} \in \mathcal{B}$. Since $Pw_0 = w_0$ and $Tp_{ij} = p_{ij}T$, we have

$$P(Tv_1 \oplus \cdots \oplus Tv_n) = Tv_1 \oplus \cdots \oplus Tv_n \in M .$$

Hence by definition of M there exists $S \in \mathcal{A}$ such that

$$Sv_1 \oplus \cdots \oplus Sv_n = Tv_1 \oplus \cdots \oplus Tv_n .$$

This proves that $T = S$, so $T \in \mathcal{A}$. □

4.1.6 Isotypic Decomposition and Multiplicities

Let $\mathcal{A}$ be an associative algebra with unit 1. If U is a finite-dimensional irreducible $\mathcal{A}$-module, we denote by $[U]$ the equivalence class of all $\mathcal{A}$-modules equivalent to U. Let $\widehat{\mathcal{A}}$ be the set of all equivalence classes of finite-dimensional irreducible $\mathcal{A}$-modules. Suppose that V is an $\mathcal{A}$-module (we do not assume that V is finite-dimensional). For each $\lambda \in \widehat{\mathcal{A}}$ we define the λ*-isotypic subspace*

$$V_{(\lambda)} = \sum_{U \subset V,\, [U] = \lambda} U .$$

Fix a module F^λ in the class λ for each $\lambda \in \widehat{\mathcal{A}}$. There is a tautological linear map

$$S_\lambda : \mathrm{Hom}_{\mathcal{A}}(F^\lambda, V) \otimes F^\lambda \longrightarrow V , \qquad S_\lambda(u \otimes w) = u(w) . \tag{4.5}$$

Make $\mathrm{Hom}_{\mathcal{A}}(F^\lambda, V) \otimes F^\lambda$ into an $\mathcal{A}$-module with action $x \cdot (u \otimes w) = u \otimes (xw)$ for $x \in \mathcal{A}$. Then S_λ is an $\mathcal{A}$-intertwining map. If $0 \neq u \in \mathrm{Hom}_{\mathcal{A}}(F^\lambda, V)$ then Schur's lemma (Lemma 4.1.4) implies that $u(F^\lambda)$ is an irreducible $\mathcal{A}$-submodule of V isomorphic to F^λ. Hence

$$S_\lambda\left(\mathrm{Hom}_{\mathcal{A}}(F^\lambda, V) \otimes F^\lambda\right) \subset V_{(\lambda)} \quad \text{for every } \lambda \in \widehat{\mathcal{A}}.$$

Definition 4.1.14. The $\mathcal{A}$-module V is *locally completely reducible* if the cyclic $\mathcal{A}$-submodule $\mathcal{A}v$ is finite-dimensional and completely reducible for every $v \in V$.

For example, if G is a reductive linear algebraic group, then by Proposition 1.4.4 $\mathcal{O}[G]$ is a locally completely reducible module for the group algebra $\mathcal{A}[G]$ relative to the left or right translation action of G.

Proposition 4.1.15. *Let V be a locally completely reducible $\mathcal{A}$-module. Then the map S_λ gives an $\mathcal{A}$-module isomorphism* $\mathrm{Hom}_{\mathcal{A}}(F^\lambda, V) \otimes F^\lambda \cong V_{(\lambda)}$ *for each* $\lambda \in \widehat{\mathcal{A}}$. *Furthermore,*

$$V = \bigoplus_{\lambda \in \widehat{\mathcal{A}}} V_{(\lambda)} \quad \textit{(algebraic direct sum)}. \tag{4.6}$$

Proof. If $U \subset V$ is an $\mathcal{A}$-invariant finite-dimensional irreducible subspace with $[U] = \lambda$, then there exists $u \in \mathrm{Hom}_{\mathcal{A}}(F^\lambda, V)$ such that $\mathrm{Range}(u) = U$. Hence S_λ is surjective.

To show that S_λ is injective, let $u_i \in \mathrm{Hom}_{\mathcal{A}}(F^\lambda, V)$ and $w_i \in F^\lambda$ for $i = 1, \ldots, k$, and suppose that $\sum_i u_i(w_i) = 0$. We may assume that $\{w_1, \ldots, w_k\}$ is linearly independent and that $u_i \neq 0$ for all i. Let $W = u_1(F^\lambda) + \cdots + u_k(F^\lambda)$. Then W is a finite-dimensional $\mathcal{A}$-submodule of $V_{(\lambda)}$; hence by Proposition 4.1.11, $W = W_1 \oplus \cdots \oplus W_m$ with W_j irreducible and $[W_j] = \lambda$. Let $\varphi_j : W \longrightarrow F^\lambda$ be the projection onto the subspace W_j followed by an $\mathcal{A}$-module isomorphism with F^λ. Then $\varphi_j \circ u_i \in \mathrm{End}_{\mathcal{A}}(F^\lambda)$. Thus $\varphi_j \circ u_i = c_{ij} I$ with $c_{ij} \in \mathbb{C}$ (Schur's lemma), and we have

$$0 = \sum_i \varphi_j u_i(w_i) = \sum_i c_{ij} w_i \quad \text{for } j = 1, \ldots, m.$$

Since $\{w_1, \ldots, w_k\}$ is linearly independent, we conclude that $c_{ij} = 0$. Hence the projection of $\mathrm{Range}(u_i)$ onto W_j is zero for $j = 1, \ldots, m$. This implies that $u_i = 0$, proving injectivity of S_λ.

The definition of local complete reducibility implies that V is spanned by the spaces $V_{(\lambda)}$ ($\lambda \in \widehat{\mathcal{A}}$). So it remains to prove only that these spaces are linearly independent. Fix distinct classes $\{\lambda_1, \ldots, \lambda_d\} \subset \widehat{\mathcal{A}}$ such that $V_{(\lambda_i)} \neq \{0\}$. We will prove by induction on d that the sum $E = V_{(\lambda_1)} + \cdots + V_{(\lambda_d)}$ is direct. If $d = 1$ there is nothing to prove. Let $d > 1$ and assume that the result holds for $d-1$ summands. Set $U = V_{(\lambda_1)} + \cdots + V_{(\lambda_{d-1})}$. Then $E = U + V_{(\lambda_d)}$ and $U = V_{(\lambda_1)} \oplus \cdots \oplus V_{(\lambda_{d-1})}$ by the induction hypothesis. For $i < d$ let $p_i : U \longrightarrow V_{(\lambda_i)}$ be the projection corresponding to this direct sum decomposition. Suppose, for the sake of contradiction, that there exists a nonzero vector $v \in U \cap V_{(\lambda_d)}$. The $\mathcal{A}$-submodule $\mathcal{A}v$ of $V_{(\lambda_d)}$ is then nonzero, finite-dimensional, and completely reducible. Hence by the last part of Proposition 4.1.11 there is a decomposition

$$\mathcal{A}v = W_1 \oplus \cdots \oplus W_r \quad \text{with} \quad [W_i] = \lambda_d \ . \tag{4.7}$$

On the other hand, since $\mathcal{A}v \subset U$, there must exist an $i < d$ such that $p_i(\mathcal{A}v)$ is nonzero. But Proposition 4.1.11 then implies that $p_i(\mathcal{A}v)$ is a direct sum of irreducible modules of type λ_i. Since $\lambda_i \neq \lambda_d$, this contradicts (4.7), by Schur's lemma. Hence $U \cap V_{(\lambda_d)} = (0)$, and we have $E = V_{(\lambda_1)} \oplus \cdots \oplus V_{(\lambda_d)}$. □

We call (4.6) the *primary decomposition* of V. We set

$$m_V(\lambda) = \dim \operatorname{Hom}_{\mathcal{A}}(F^\lambda, V) \quad \text{for } \lambda \in \widehat{\mathcal{A}} \ ,$$

and call $m_V(\lambda)$ the *multiplicity* of ξ in V. The multiplicities may be finite or infinite; likewise for the number of nonzero summands in the primary decomposition. We call the set

$$\operatorname{Spec}(V) = \{\lambda \in \widehat{\mathcal{A}} : m_V(\lambda) \neq 0\}$$

the *spectrum* of the $\mathcal{A}$-module V. The primary decomposition of V gives an isomorphism

$$V \cong \bigoplus_{\lambda \in \operatorname{Spec}(V)} \operatorname{Hom}_{\mathcal{A}}(F^\lambda, V) \otimes F^\lambda \tag{4.8}$$

with the action of $\mathcal{A}$ only on the second factor in each summand.

Assume that V is completely reducible under $\mathcal{A}$. The primary decomposition has a finite number of summands, since V is finite-dimensional, and the multiplicities are finite. We claim that $m_V(\lambda)$ is also given by

$$m_V(\lambda) = \dim \operatorname{Hom}_{\mathcal{A}}(V, F^\lambda) \ . \tag{4.9}$$

To prove this, let $m = m_V(\lambda)$. Then $V = W \oplus (F^\lambda)^{(m)}$, where W is the sum of the isotypic subspaces for representations not equivalent to λ. If $T \in \operatorname{Hom}_{\mathcal{A}}(V, F^\lambda)$, then by Schur's lemma $T(W) = 0$ and T is a linear combination of the operators $\{T_1, \ldots, T_m\}$, where

$$T_i(w \oplus v_1 \oplus \cdots \oplus v_m) = v_i \quad \text{for } w \in W \text{ and } v_i \in V \ .$$

These operators are linearly independent, so they furnish a basis for $\operatorname{Hom}_{\mathcal{A}}(V, F^\lambda)$.

Remark 4.1.16. Let U and V be completely reducible $\mathcal{A}$-modules. Define

$$\langle U, V \rangle = \dim \operatorname{Hom}_{\mathcal{A}}(U, V) \ .$$

Then from Proposition 4.1.15 we have

$$\langle U, V \rangle = \sum_{\lambda \in \widehat{\mathcal{A}}} m_U(\lambda)\, m_V(\lambda) \ . \tag{4.10}$$

It follows that $\langle U,V\rangle = \langle V,U\rangle$ and $\langle U,V\oplus W\rangle = \langle U,V\rangle + \langle U,W\rangle$ for any completely reducible $\mathcal{A}$-modules U, V, and W.

4.1.7 Characters

Let $\mathcal{A}$ be an associative algebra with unit 1. If (ρ,V) is a finite-dimensional representation of $\mathcal{A}$, then the *character* of the representation is the linear functional $\mathrm{ch}V$ on $\mathcal{A}$ given by

$$\mathrm{ch}V(a) = \mathrm{tr}_V(\rho(a)) \qquad \text{for } a \in \mathcal{A}\,.$$

Proposition 4.1.17. *Characters satisfy* $\mathrm{ch}V(ab) = \mathrm{ch}V(ba)$ *for all* $a,b \in \mathcal{A}$ *and* $\mathrm{ch}V(1) = \dim V$. *Furthermore, if* $U \subset V$ *is a submodule and* $W = V/U$, *then* $\mathrm{ch}V = \mathrm{ch}U + \mathrm{ch}W$.

Proof. The first two properties are obvious from the definition. The third follows by picking a subspace $Z \subset V$ complementary to U. Then the matrix of $\rho(a), a \in \mathcal{A}$, is in block triangular form relative to the decomposition $V = U \oplus Z$, and the diagonal blocks give the action of a on U and on V/U. □

The use of characters in representation theory is a powerful tool (similar to the use of generating functions in combinatorics). This will become apparent in later chapters. Let us find the extent to which a representation is determined by its character.

Lemma 4.1.18. *Suppose* $(\rho_1,V_1),\ldots,(\rho_r,V_r)$ *are finite-dimensional irreducible representations of* $\mathcal{A}$ *such that* ρ_i *is not equivalent to* ρ_j *when* $i \neq j$. *Then the set* $\{\mathrm{ch}V_1,\ldots,\mathrm{ch}V_r\}$ *of linear functionals on* $\mathcal{A}$ *is linearly independent.*

Proof. Set $V = V_1 \oplus \cdots \oplus V_r$ and $\rho = \rho_1 \oplus \cdots \oplus \rho_r$. Then (ρ,V) is a completely reducible representation of $\mathcal{A}$ by Proposition 4.1.11. Let $\mathcal{B}$ be the commutant of $\rho(\mathcal{A})$. Since the representations are irreducible and mutually inequivalent, Schur's lemma (Lemma 4.1.4) implies that the elements of $\mathcal{B}$ preserve each subspace V_j and act on it by scalars. Hence by the double commutant theorem (Theorem 4.1.13),

$$\rho(\mathcal{A}) = \mathrm{End}(V_1) \oplus \cdots \oplus \mathrm{End}(V_r)\,.$$

Let $I_i \in \mathrm{End}(V_i)$ be the identity operator on V_i. For each i there exists $Q_i \in \mathcal{A}$ with $\rho(Q_i)|_{V_j} = \delta_{ij} I_j$. We have

$$\mathrm{ch}V_j(Q_i) = \delta_{ij} \dim V_i\,.$$

Thus given a linear relation $\sum a_j\, \mathrm{ch}V_j = 0$, we may evaluate on Q_i to conclude that $a_i \dim V_i = 0$. Hence $a_i = 0$ for all i. □

Let (ρ,V) be a finite-dimensional $\mathcal{A}$-module. A *composition series* (or *Jordan–Hölder series*) for V is a sequence of submodules

$$(0) = V_0 \subset V_1 \subset \cdots \subset V_r = V$$

such that $0 \neq W_i = V_i/V_{i-1}$ is irreducible for $i = 1, \ldots, r$. It is clear by induction on $\dim V$ that a composition series always exists. We define the *semisimplification* of V to be the module

$$V_{ss} = \bigoplus_{i=1}^{r} W_i \, .$$

By (3) of Proposition 4.1.17 and the obvious induction, we see that

$$\operatorname{ch} V = \sum_{i=1}^{r} \operatorname{ch}(V_i/V_{i-1}) = \operatorname{ch} V_{ss} \, . \tag{4.11}$$

Theorem 4.1.19. *Let (ρ, V) be a finite-dimensional $\mathcal{A}$-module. The irreducible factors in a composition series for V are unique up to isomorphism and order of appearance. Furthermore, the module V_{ss} is uniquely determined by* $\operatorname{ch} V$ *up to isomorphism. In particular, if V is completely reducible, then V is uniquely determined up to isomorphism by* $\operatorname{ch} V$.

Proof. Let (ρ_i, U_i), for $i = 1, \ldots, n$, be the pairwise inequivalent irreducible representations that occur in the composition series for V, with corresponding multiplicities m_i. Then

$$\operatorname{ch} V = \sum_{i=1}^{n} m_i \operatorname{ch} U_i$$

by (4.11). Lemma 4.1.18 implies that the multiplicities m_i are uniquely determined by $\operatorname{ch} V$. □

Example

Let $G = \mathbf{SL}(2, \mathbb{C})$ and let (ρ, V) be a regular representation of G. Let

$$d(q) = \operatorname{diag}[q, q^{-1}] \quad \text{for } q \in \mathbb{C}^{\times} \, .$$

If $g \in G$ and $\operatorname{tr}(g)^2 \neq 4$, then $g = hd(q)h^{-1}$ for some $h \in G$ (see Exercises 1.6.4, #7), where the eigenvalues of g are $\{q, q^{-1}\}$. Hence $\operatorname{ch} V(g) = \operatorname{ch} V(d(q))$. Since the function $g \mapsto \operatorname{ch} V(\rho(g))$ is regular and G is connected (Theorem 2.2.5), the character is determined by its restriction to the set $\{g \in G : \operatorname{tr}(g)^2 \neq 4\}$. Hence $\operatorname{ch} V$ is uniquely determined by the function $q \mapsto \operatorname{ch} V(d(q))$ for $q \in \mathbb{C}^{\times}$. Furthermore, since $d(q)$ is conjugate to $d(q^{-1})$ in G, this function on $\mathbb{C}^{\times}$ is invariant under the symmetry $q \mapsto q^{-1}$ arising from the action of the Weyl group of G on the diagonal matrices.

Let $(\rho_k, F^{(k)})$ be the $(k+1)$-dimensional irreducible representation of $\mathbf{SL}(2, \mathbb{C})$ (see Proposition 2.3.5). Then

$$\operatorname{ch} F^{(k)}(d(q)) = q^k + q^{k-2} + \cdots + q^{-k+2} + q^{-k}$$

(note the invariance under $q \mapsto q^{-1}$). For a positive integer n we define

$$[n]_q = q^{n-1} + q^{n-3} + \cdots + q^{-n+3} + q^{-n+1} = \frac{q^n - q^{-n}}{q - q^{-1}}$$

as a rational function of q. Then we can write $\operatorname{ch} F^{(k)}(d(q)) = [k+1]_q$.

Define $[0]_q = 1$, $[n]_q! = \prod_{j=0}^{n}[n-j]_q$ (the *q-factorial*), and

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_q = \frac{[m+n]_q!}{[m]_q![n]_q!} . \qquad (q\text{-binomial coefficient})$$

Theorem 4.1.20 (Hermite Reciprocity). *Let $S^j(F^{(k)})$ be the jth symmetric power of $F^{(k)}$. Then for $q \in \mathbb{C}^\times$,*

$$\operatorname{ch} S^j(F^{(k)})(d(q)) = \begin{bmatrix} k+j \\ k \end{bmatrix}_q . \tag{4.12}$$

In particular, $S^j(F^{(k)}) \cong S^k(F^{(j)})$ as representations of $\mathbf{SL}(2,\mathbb{C})$.

To prove this theorem we need some further character identities. Fix k and write $f_j(q) = \operatorname{ch} S^j(F^{(k)})(d(q))$ for $q \in \mathbb{C}^\times$. Let $\{x_0, \ldots, x_k\}$ be a basis for $F^{(k)}$ such that

$$\rho_k(d(q))x_j = q^{k-2j}x_j .$$

Then the monomials $x_0^{m_0}x_1^{m_1}\cdots x_k^{m_k}$ with $m_0 + \cdots + m_k = j$ give a basis for $S^j(F^{(k)})$, and $d(q)$ acts on such a monomial by the scalar q^r, where

$$r = km_0 + (k-2)m_1 + \cdots + (2-k)m_{k-1} - km_k .$$

Hence

$$f_j(q) = \sum_{m_0,\ldots,m_k} q^{km_0+(k-2)m_1+\cdots+(2-k)m_{k-1}-km_k}$$

with the sum over all nonnegative integers $m_0, \ldots, m_k$ such that $m_0 + \cdots + m_k = j$.

We form the *generating function*

$$f(t,q) = \sum_{j=0}^{\infty} t^j f_j(q) ,$$

which we view as a formal power series in the indeterminate t with coefficients in the ring $\mathbb{C}(q)$ of rational functions of q. If we let $\mathbb{C}^\times$ act by scalar multiplication on $F^{(k)}$, then $t \in \mathbb{C}^\times$ acts by multiplication by t^j on $S_j(F^{(k)})$ and this action commutes with the action of $\mathbf{SL}(2,\mathbb{C})$. Thus we can also view $f(t,q)$ as a *formal character* for the joint action of $\mathbb{C}^\times \times \mathbf{SL}(2,\mathbb{C})$ on the infinite-dimensional graded vector space $S(F^{(k)})$.

Lemma 4.1.21. *The generating function factors as*

$$f(t,q) = \prod_{j=0}^{k} (1 - tq^{k-2j})^{-1}. \tag{4.13}$$

Proof. By definition $(1 - tq^{k-2j})^{-1}$ is the formal power series

$$\sum_{m=0}^{\infty} t^m q^{m(k-2j)}. \tag{4.14}$$

Hence the right side of (4.13) is

$$\sum_{m_0,\ldots,m_k} t^{m_0+\cdots+m_k} q^{km_0+(k-2)m_1+\cdots+(2-k)m_{k-1}-km_k}$$

with the sum over all nonnegative integers $m_0,\ldots,m_k$. Thus the coefficient of t^j is $f_j(q)$. □

Since the representation $S^j(F^{(k)})$ is completely reducible, it is determined up to equivalence by its character, by Theorem 4.1.19. Just as for the ordinary binomial coefficients, one has the symmetry

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_q = \begin{bmatrix} m+n \\ m \end{bmatrix}_q.$$

To complete the proof of Theorem 4.1.20, it thus suffices to prove the following result:

Lemma 4.1.22. *One has the formal power series identity*

$$\prod_{j=0}^{k} (1 - tq^{k-2j})^{-1} = \sum_{j=0}^{\infty} t^j \begin{bmatrix} k+j \\ k \end{bmatrix}_q,$$

where the factors on the left side are defined by (4.14).

Proof. The proof proceeds by induction on k. The case $k = 0$ is the formal power series identity $(1-t)^{-1} = \sum_{j=0}^{\infty} t^j$. Now set

$$H_k(t,q) = \sum_{j=0}^{\infty} t^j \begin{bmatrix} k+j \\ k \end{bmatrix}_q,$$

and assume that

$$H_k(t,q) = \prod_{j=0}^{k} (1 - tq^{k-2j})^{-1}.$$

It is easy to check that the q-binomial coefficients satisfy the recurrence

$$\begin{bmatrix} k+1+j \\ k+1 \end{bmatrix}_q = \frac{q^{k+1+j} - q^{-k-1-j}}{q^{k+1} - q^{-k-1}} \begin{bmatrix} k+j \\ k \end{bmatrix}_q .$$

Thus

$$H_{k+1}(t,q) = \frac{q^{k+1}}{q^{k+1} - q^{-k-1}} H_k(tq,q) - \frac{q^{-k-1}}{q^{k+1} - q^{-k-1}} H_k(tq^{-1},q) .$$

Hence by the induction hypothesis we have

$$\begin{aligned} H_{k+1}(t,q) &= \frac{q^{k+1}}{(q^{k+1} - q^{-k-1}) \prod_{j=0}^{k} (1 - tq^{k+1-2j})} \\ &\qquad - \frac{q^{-k-1}}{(q^{k+1} - q^{-k-1}) \prod_{j=0}^{k} (1 - tq^{k-1-2j})} \\ &= \Big(\frac{q^{k+1}}{1 - tq^{k+1}} - \frac{q^{-k-1}}{1 - tq^{-k-1}} \Big) \Big/ \Big((q^{k+1} - q^{-k-1}) \prod_{j=1}^{k} (1 - tq^{k+1-2j}) \Big) \\ &= \prod_{j=0}^{k+1} (1 - tq^{k+1-2j})^{-1} . \end{aligned}$$

□

4.1.8 Exercises

1. Let $\mathcal{A}$ be an associative algebra over $\mathbb{C}$ with unit element 1. Then $\mathcal{A} \otimes \mathcal{A}$ is an associative algebra with unit element $1 \otimes 1$, where the multiplication is defined by $(a \otimes b)(c \otimes d) = (ac) \otimes (bc)$ on decomposable tensors and extended to be bilinear. A *bialgebra structure* on $\mathcal{A}$ consists of an algebra homomorphism $\Delta : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$ (called the *comultiplication*) and an algebra homomorphism $\varepsilon : \mathcal{A} \longrightarrow \mathbb{C}$ (called the *counit*) that satisfy the following:

   (coassociativity) The maps $\Delta \otimes I_{\mathcal{A}}$ and $I_{\mathcal{A}} \otimes \Delta$ from $\mathcal{A}$ to $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ coincide: $(\Delta \otimes I_{\mathcal{A}})(\Delta(a)) = (I_{\mathcal{A}} \otimes \Delta)(\Delta(a))$ for all $a \in \mathcal{A}$, where $(\mathcal{A} \otimes \mathcal{A}) \otimes \mathcal{A}$ is identified with $\mathcal{A} \otimes (\mathcal{A} \otimes \mathcal{A})$ as usual and $I_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}$ is the identity map.

   (counit) The maps $(I_{\mathcal{A}} \otimes \varepsilon) \circ \Delta$ and $(\varepsilon \otimes I_{\mathcal{A}}) \circ \Delta$ from $\mathcal{A}$ to $\mathcal{A}$ coincide: $(I_{\mathcal{A}} \otimes \varepsilon)(\Delta(a)) = (\varepsilon \otimes I_{\mathcal{A}})(\Delta(a))$ for all $a \in \mathcal{A}$, where we identify $\mathbb{C} \otimes \mathcal{A}$ with $\mathcal{A}$ as usual.

   (a) Let G be a group and let $\mathcal{A} = \mathcal{A}[G]$ with convolution product. Define Δ and ε on the basis elements δ_x for $x \in G$ by $\Delta(\delta_x) = \delta_x \otimes \delta_x$ and $\varepsilon(\delta_x) = 1$, and extend these maps by linearity. Show that Δ and ε satisfy the conditions for a bialgebra structure on $\mathcal{A}$ and that $\langle \Delta(f), g \otimes h \rangle = \langle f, gh \rangle$ for $f, g, h \in \mathcal{A}[G]$. Here we write $\langle \phi, \psi \rangle = \sum_{x \in X} \phi(x) \psi(x)$ for complex-valued functions ϕ, ψ on a set X, and gh denotes the pointwise product of the functions g and h.

(b) Let G be a group and consider $\mathcal{A}[G]$ as the commutative algebra of $\mathbb{C}$-valued functions on G with pointwise multiplication and the constant function 1 as identity element. Identify $\mathcal{A}[G] \otimes \mathcal{A}[G]$ with $\mathcal{A}[G \times G]$ by $\delta_x \otimes \delta_y \longleftrightarrow \delta_{(x,y)}$ for $x, y \in G$. Define Δ by $\Delta(f)(x,y) = f(xy)$ and define $\varepsilon(f) = f(1)$, where $1 \in G$ is the identity element. Show that this defines a bialgebra structure on $\mathcal{A}[G]$ and that $\langle \Delta(f), g \otimes h \rangle = \langle f, g * h \rangle$ for $f, g, h \in \mathcal{A}[G]$, where $\langle \phi, \psi \rangle$ is defined as in (a) and where $g * h$ denotes the convolution product of the functions g and h.

(c) Let G be a linear algebraic group. Consider $\mathcal{A} = \mathcal{O}[G]$ as a (commutative) algebra with pointwise multiplication of functions and the constant function 1 as the identity element. Identify $\mathcal{A} \otimes \mathcal{A}$ with $\mathcal{O}[G \times G]$ as in Proposition 1.4.4 and define Δ and ε by the same formulas as in (b). Show that this defines a bialgebra structure on $\mathcal{A}$.

(d) Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$ and let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Define $\Delta(X) = X \otimes 1 + 1 \otimes X$ for $X \in \mathfrak{g}$. Show that $\Delta([X,Y]) = \Delta(X)\Delta(Y) - \Delta(Y)\Delta(X)$, and conclude that Δ extends uniquely to an algebra homomorphism $\Delta : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$. Let $\varepsilon : U(\mathfrak{g}) \longrightarrow \mathbb{C}$ be the unique algebra homomorphism such that $\varepsilon(X) = 0$ for all $X \in \mathfrak{g}$, as in Section 3.2.1. Show that Δ and ε define a bialgebra structure on $U(\mathfrak{g})$.

(e) Suppose G is a linear algebraic group with Lie algebra $\mathfrak{g}$. Define a bilinear form on $U(\mathfrak{g}) \times \mathcal{O}[G]$ by $\langle T, f \rangle = Tf(I)$ for $T \in U(\mathfrak{g})$ and $f \in \mathcal{O}[G]$, where the action of $U(\mathfrak{g})$ on $\mathcal{O}[G]$ comes from the action of $\mathfrak{g}$ as left-invariant vector fields. Show that $\langle \Delta(T), f \otimes g \rangle = \langle T, fg \rangle$ for all $T \in U(\mathfrak{g})$ and $f, g \in \mathcal{O}[G]$, where Δ is defined as in (d). (This shows that the comultiplication on $U(\mathfrak{g})$ is dual to the pointwise multiplication on $\mathcal{O}[G]$.)

2. Let $\mathcal{A}$ be an associative algebra over $\mathbb{C}$, and suppose Δ and ε give $\mathcal{A}$ the structure of a bialgebra, in the sense of the previous exercise. Let (V, ρ) and (W, σ) be representations of $\mathcal{A}$.

(a) Show that the map $(a,b) \mapsto \rho(a) \otimes \sigma(b)$ extends to a representation of $\mathcal{A} \otimes \mathcal{A}$ on $V \otimes W$, denoted by $\rho \widehat{\otimes} \sigma$.

(b) Define $(\rho \otimes \sigma)(a) = (\rho \widehat{\otimes} \sigma)(\Delta(a))$ for $a \in \mathcal{A}$. Show that $\rho \otimes \sigma$ is a representation of $\mathcal{A}$, called the *tensor product* $\rho \otimes \sigma$ of the representations ρ and σ.

(c) When $\mathcal{A}$ and Δ are given as in (a) or (d) of the previous exercise, verify that the tensor product defined via the map Δ is the same as the tensor product defined in Section 4.1.1.

3. Let $\mathcal{A}$ be a bialgebra, in the sense of the previous exercises with comultiplication map Δ and counit ε. Let $S : \mathcal{A} \longrightarrow \mathcal{A}$ be an *antiautomorphsim* ($S(xy) = S(y)S(x)$ for all $x, y \in \mathcal{A}$). Then S is called an *antipode* if $\mu((S \otimes I_{\mathcal{A}})(\Delta(a))) = \varepsilon(a)1$ and $\mu((I_{\mathcal{A}} \otimes S)(\Delta(a))) = \varepsilon(a)1$ for all $a \in \mathcal{A}$, where $\mu : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ is the multiplication map. A bialgebra with an antipode is called a *Hopf algebra*.

(a) Let G be a group, and let $\mathcal{A} = \mathcal{A}[G]$ with convolution multiplication. Let Δ and ε be defined as in the exercise above, and let $Sf(x) = f(x^{-1})$ for $f \in \mathcal{A}[G]$ and $x \in G$. Show that S is an antipode.

(b) Let G be a group, and let $\mathcal{A} = \mathcal{A}[G]$ with pointwise multiplication. Let Δ and ε be defined as in the exercise above, and let $Sf(x) = f(x^{-1})$ for $f \in \mathcal{A}[G]$ and

$x \in G$. Show that S is an antipode (the same holds when G is a linear algebraic group and $A = \mathcal{O}[G]$).
(c) Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$. Define the maps Δ and ε on $U(\mathfrak{g})$ as in the exercise above. Let $S(X) = -X$ for X in $\mathfrak{g}$. Show that S extends to an antiautomorphism of $U(\mathfrak{g})$ and satisfies the conditions for an antipode.

4. Let A be a Hopf algebra over $\mathbb{C}$ with antipode S.
(a) Given a representation (ρ, V) of A, define $\rho^S(x) = \rho(Sx)^*$ for $x \in A$. Show that (ρ^S, V^*) is a representation of A.
(b) Show that the representation (ρ^S, V^*) is the *dual* representation to (ρ, V) when A is either $A[G]$ with convolution multiplication or $U(\mathfrak{g})$ (where $\mathfrak{g}$ is a Lie algebra) and the antipode is defined as in the exercise above.
5. Let $A = A[x]$ and let $T \in M_n[\mathbb{C}]$. Define a representation ρ of A on $\mathbb{C}^n$ by $\rho(x) = T$. When is the representation ρ completely reducible? (HINT: Put T into Jordan canonical form.)
6. Let A be an associative algebra and let V be a completely reducible finite-dimensional A-module.
(a) Show that V is irreducible if and only if $\dim \mathrm{Hom}_A(V,V) = 1$.
(b) Does (a) hold if V is not completely reducible? (HINT: Consider the algebra of all upper-triangular 2×2 matrices.)
7. Let (ρ, V) and (σ, W) be finite-dimensional representations of a group G and let $g \in G$.
(a) Show that $\mathrm{ch}(V \otimes W)(g) = \mathrm{ch}V(g) \cdot \mathrm{ch}W(g)$.
(b) Show that $\mathrm{ch}(\bigwedge^2 V)(g) = \frac{1}{2}\Big((\mathrm{ch}V(g))^2 - \mathrm{ch}V(g^2) \Big)$.
(c) Show that $\mathrm{ch}(S^2(V))(g) = \frac{1}{2}\Big((\mathrm{ch}V(g))^2 + \mathrm{ch}V(g^2) \Big)$.
(HINT: Let $\{\lambda_i\}$ be the eigenvalues of $\rho(g)$ on V. Then $\{\lambda_i\lambda_j\}_{i<j}$ are the eigenvalues of g on $\bigwedge^2 V$ and $\{\lambda_i\lambda_j\}_{i\le j}$ are the eigenvalues of g on $S^2(V)$.)

The following exercises use the notation in Section 4.1.7.

8. Let (σ, W) be a regular representation of $\mathbf{SL}(2,\mathbb{C})$. For $q \in \mathbb{C}^\times$ let $f(q) = \mathrm{ch}W(d(q))$. Write $f(q) = f_{\mathrm{even}}(q) + f_{\mathrm{odd}}(q)$, where $f_{\mathrm{even}}(-q) = f_{\mathrm{even}}(q)$ and $f_{\mathrm{odd}}(-q) = -f_{\mathrm{odd}}(q)$.
(a) Show that $f_{\mathrm{even}}(q) = f_{\mathrm{even}}(q^{-1})$ and $f_{\mathrm{odd}}(q) = f_{\mathrm{odd}}(q^{-1})$.
(b) Let $f_{\mathrm{even}}(q) = \sum_{k\in\mathbb{Z}} a_k q^{2k}$ and $f_{\mathrm{odd}}(q) = \sum_{k\in\mathbb{Z}} b_k q^{2k+1}$. Show that the sequences $\{a_k\}$ and $\{b_k\}$ are unimodal. (HINT: See Exercises 2.3.4 #6.)
9. Let (σ, W) be a regular representation of $\mathbf{SL}(2,\mathbb{C})$ and let $W \cong \bigoplus_{k\ge 0} m_k F^{(k)}$ be the decomposition of W into isotypic components. We say that W is *even* if $m_k = 0$ for all odd integers k, and we say that W is *odd* if $m_k = 0$ for all even integers.
(a) Show W is even if and only if $\mathrm{ch}W(d(-q)) = \mathrm{ch}W(d(q))$, and odd if and only if $\mathrm{ch}W(d(-q)) = -\mathrm{ch}W(d(q))$. (HINT: Use Proposition 2.3.5.)
(b) Show that $S^j(F^{(k)})$ is even if jk is even and odd if jk is odd. (HINT: Use the model for $F^{(k)}$ from Section 2.3.2 to show that $-I \in \mathbf{SL}(2,\mathbb{C})$ acts on $F^{(k)}$ by $(-1)^k$ and hence acts by $(-1)^{jk}$ on $S^j(F^{(k)})$.)

10. Set $f(q) = \left[{m+n \atop m}\right]_q$ for $q \in \mathbb{C}^\times$ and positive integers m and n.
(a) Show that $f(q) = f(q^{-1})$.
(b) Show that $f(q) = \sum_{k\in\mathbb{Z}} a_k q^{2k+\varepsilon}$, where $\varepsilon = 0$ when mn is even and $\varepsilon = 1$ when mn is odd.
(c) Show that the sequence $\{a_k\}$ in (b) is unimodal. (HINT: Use the previous exercises and Theorem 4.1.20.)

11. (a) Show (by a computer algebra system or otherwise) that

$$\left[{4+3 \atop 3}\right]_q = q^{12} + q^{10} + 2q^8 + 3q^6 + 4q^4 + 4q^2 + 5 + \cdots$$

(where $\cdots$ indicates terms in negative powers of q).
(b) Use (a) to prove that $S^3(V_4) \cong S^4(V_3) \cong V_{12} \oplus V_8 \oplus V_6 \oplus V_4 \oplus V_0$.
(HINT: Use Proposition 2.3.5 and Theorem 4.1.20.)

12. (a) Show (by a computer algebra system or otherwise) that

$$\left[{5+3 \atop 3}\right]_q = q^{15} + q^{13} + 2q^{11} + 3q^9 + 4q^7 + 5q^5 + 6q^3 + 6q + \cdots$$

(where $\cdots$ indicates terms in negative powers of q).
(b) Use (a) to prove that $S^3(V_5) \cong S^5(V_3) \cong V_{15} \oplus V_{11} \oplus V_9 \oplus V_7 \oplus V_5 \oplus V_3$.
(HINT: Use Proposition 2.3.5 and Theorem 4.1.20.)

13. For $n \in \mathbb{N}$ and $q \in \mathbb{C}$ define $\{n\}_1 = n$ and

$$\{n\}_q = q^{n-1} + q^{n-2} + \cdots + q + 1 = (q^n - 1)/(q-1) \quad \text{for } q \neq 1 .$$

(a) Show that $\{n\}_{q^2} = q^{n-1}[n]_q$.
(b) Define

$$C_{n+m,m}(q) = \frac{\{m+n\}_q!}{\{m\}_q!\{n\}_q!} .$$

(This is an alternative version of the q-binomial coefficient that also gives the ordinary binomial coefficient when specialized to $q = 1$.) Let p be a prime and let $\mathbb{F}$ be the field with $q = p^n$ elements. Prove that $C_{m+n,m}(q)$ is the number of m-dimensional subspaces in the vector space $\mathbb{F}^{m+n}$. (HINT: The number of nonzero elements of $\mathbb{F}^{m+n}$ is $q^{n+m} - 1$. If $v \in \mathbb{F}^{m+n} - \{0\}$ then the number of elements that are not multiples of v is $q^{n+m} - q$. Continuing in this way we find that the cardinality of the set of all linearly independent m-tuples $\{v_1, \ldots, v_m\}$ is $(q^{n+m} - 1)(q^{n+m-1} - 1)\cdots(q^{n+1} - 1) = a_{n,m}$. The desired cardinality is thus $a_{n,m}/a_{0,m} = C_{n+m,m}(q)$.)

4.2 Duality for Group Representations

In this section we prove the duality theorem. As first applications we obtain *Schur–Weyl duality* for representations of $\mathbf{GL}(n,\mathbb{C})$ on tensor spaces and the decomposition of the representation of $G\times G$ on $\mathcal{O}[G]$. Further applications of the duality theorem will occur in later chapters.

4.2.1 General Duality Theorem

Assume that $G \subset \mathbf{GL}(n,\mathbb{C})$ is a reductive linear algebraic group. Let $\widehat{G}$ denote the equivalence classes of irreducible regular representations of G and fix a representation (π^λ, F^λ) in the class λ for each $\lambda \in \widehat{G}$. We view representation spaces for G as modules for the group algebra $\mathcal{A}[G]$, as in Section 4.1.1, and identify $\widehat{G}$ with a subset of $\widehat{\mathcal{A}[G]}$.

Let (ρ, L) be a locally regular representation of G with $\dim L$ countable. Then ρ is a locally completely reducible representation of $\mathcal{A}[G]$, and the irreducible $\mathcal{A}[G]$ submodules of L are irreducible regular representations of G (since G is reductive). Thus the nonzero isotypic components $L_{(\lambda)}$ are labeled by $\lambda \in \widehat{G}$. We shall write $\mathrm{Spec}(\rho)$ for the set of representation types that occur in the primary decomposition of L. Then by Proposition 4.1.15 we have

$$L \cong \bigoplus_{\lambda \in \mathrm{Spec}(\rho)} \mathrm{Hom}(F^\lambda, L) \otimes F^\lambda$$

as a G-module, with $g \in G$ acting by $I \otimes \pi^\lambda(g)$ on the summand of type λ. We now focus on the *multiplicity spaces* $\mathrm{Hom}(F^\lambda, L)$ in this decomposition.

Assume that $\mathcal{R} \subset \mathrm{End}(L)$ is a subalgebra that satisfies the following conditions:

(i) $\mathcal{R}$ acts irreducibly on L,
(ii) if $g \in G$ and $T \in \mathcal{R}$ then $\rho(g)T\rho(g)^{-1} \in \mathcal{R}$ (so G acts on $\mathcal{R}$), and
(iii) the representation of G on $\mathcal{R}$ in (ii) is locally regular.

If $\dim L < \infty$, the only choice for $\mathcal{R}$ is $\mathrm{End}(L)$ by Corollary 4.1.7. By contrast, when $\dim L = \infty$ there may exist many such algebras $\mathcal{R}$; we shall see an important example in Section 5.6.1 (the *Weyl algebra* of linear differential operators with polynomial coefficients).

Fix $\mathcal{R}$ satisfying the conditions (i) and (ii) and let

$$\mathcal{R}^G = \{T \in \mathcal{R} : \rho(g)T = T\rho(g) \quad \text{for all } g \in G\}$$

(the commutant of $\rho(G)$ in $\mathcal{R}$). Since G is reductive, we may view L as a locally completely irreducible representation of $\mathcal{A}[G]$. Since elements of $\mathcal{R}^G$ commute with elements of $\mathcal{A}[G]$, we have a representation of the algebra $\mathcal{R}^G \otimes \mathcal{A}[G]$ on L. The *duality theorem* describes the decomposition of this representation.

Let

$$E^\lambda = \mathrm{Hom}_G(F^\lambda, L) \quad \text{for } \lambda \in \widehat{G}\,.$$

Then E^λ is a module for $\mathcal{R}^G$ in a natural way by left multiplication, since

$$Tu(\pi^\lambda(g)v) = T\rho(g)u(v) = \rho(g)(Tu(v))$$

for $T \in \mathcal{R}^G$, $u \in E^\lambda$, $g \in G$, and $v \in F^\lambda$. Hence as a module for the algebra $\mathcal{R}^G \otimes \mathcal{A}[G]$ the space L decomposes as

$$L \cong \bigoplus_{\lambda \in \mathrm{Spec}(\rho)} E^\lambda \otimes F^\lambda\,. \tag{4.15}$$

In (4.15) an operator $T \in \mathcal{R}^G$ acts by $T \otimes I$ on the summand of type λ.

Theorem 4.2.1 (Duality). *Each multiplicity space E^λ is an irreducible $\mathcal{R}^G$ module. Furthermore, if $\lambda, \mu \in \mathrm{Spec}(\rho)$ and $E^\lambda \cong E^\mu$ as an $\mathcal{R}^G$ module, then $\lambda = \mu$.*

The duality theorem plays a central role in the representation and invariant theory of the classical groups. Here is an immediate consequence.

Corollary 4.2.2 (Duality Correspondence). *Let σ be the representation of $\mathcal{R}^G$ on L and let $\mathrm{Spec}(\sigma)$ denote the set of equivalence classes of the irreducible representations $\{E^\lambda\}$ of the algebra $\mathcal{R}^G$ that occur in L. Then the following hold:*

1. *The representation (σ, L) is a direct sum of irreducible $\mathcal{R}^G$ modules, and each irreducible submodule E^λ occurs with finite multiplicity $\dim F^\lambda$.*
2. *The map $F^\lambda \longrightarrow E^\lambda$ sets up a bijection between $\mathrm{Spec}(\rho)$ and $\mathrm{Spec}(\sigma)$.*

The proof of the duality theorem will use the following result:

Lemma 4.2.3. *Let $X \subset L$ be a finite-dimensional G-invariant subspace. Then $\mathcal{R}^G|_X = \mathrm{Hom}_G(X, L)$.*

Proof. Let $T \in \mathrm{Hom}_G(X, L)$. Then by Corollary 4.1.6 there exists $r \in \mathcal{R}$ such that $r|_X = T$. Since G is reductive, condition (iii) and Proposition 4.1.15 furnish a projection $r \mapsto r^\natural$ from $\mathcal{R}$ to $\mathcal{R}^G$. But the map $\mathcal{R} \longrightarrow \mathrm{Hom}(X, L)$ given by $y \mapsto y|_X$ intertwines the G actions, by the G invariance of X. Hence $T = T^\natural = r^\natural|_X$. □

Proof of Theorem 4.2.1. We first prove that the action of $\mathcal{R}^G$ on $\mathrm{Hom}_G(F^\lambda, L)$ is irreducible. Let $T \in \mathrm{Hom}_G(F^\lambda, L)$ be nonzero. Given another nonzero element $S \in \mathrm{Hom}_G(F^\lambda, L)$ we need to find $r \in \mathcal{R}^G$ such that $rT = S$. Let $X = TF^\lambda$ and $Y = SF^\lambda$. Then by Schur's lemma X and Y are isomorphic G-modules of class λ. Thus Lemma 4.2.3 implies that there exists $u \in \mathcal{R}^G$ such that $u|_X$ implements this isomorphism. Thus $uT : F^\lambda \longrightarrow SF^\lambda$ is a G-module isomorphism. Schur's lemma implies that there exists $c \in \mathbb{C}$ such that $cuT = S$, so we may take $r = cu$.

We now show that if $\lambda \neq \mu$ then $\mathrm{Hom}_G(F^\lambda, L)$ and $\mathrm{Hom}_G(F^\mu, L)$ are inequivalent modules for $\mathcal{R}^G$. Suppose

$$\varphi : \mathrm{Hom}_G(F^\lambda, L) \longrightarrow \mathrm{Hom}_G(F^\mu, L)$$

is an intertwining operator for the action of $\mathcal{R}^G$. Let $T \in \mathrm{Hom}_G(F^\lambda, L)$ be nonzero and set $S = \varphi(T)$. We want to show that $S = 0$. Set $U = TF^\lambda + SF^\mu$. Then since we are assuming $\lambda \neq \mu$, the sum is direct. Let $p : U \longrightarrow SF^\mu$ be the corresponding projection. Then Lemma 4.2.3 implies that there exists $r \in \mathcal{R}^G$ such that $r|_U = p$. Since $pT = 0$, we have $rT = 0$. Hence

$$0 = \varphi(rT) = r\varphi(T) = rS = pS = S ,$$

which proves that $\varphi = 0$. □

In the finite-dimensional case we can combine the duality theorem with the double commutant theorem.

Corollary 4.2.4. *Assume* $\dim L < \infty$. *Set* $\mathcal{A} = \mathrm{Span}\,\rho(G)$ *and* $\mathcal{B} = \mathrm{End}_{\mathcal{A}}(L)$. *Then* L *is a completely reducible* $\mathcal{B}$*-module. Furthermore, the following hold:*

1. *Suppose that for every* $\lambda \in \mathrm{Spec}(\rho)$ *there is given an operator* $T_\lambda \in \mathrm{End}(F^\lambda)$. *Then there exists* $T \in \mathcal{A}$ *that acts by* $I \otimes T_\lambda$ *on the* λ *summand in the decomposition* (4.15).
2. *Let* $T \in \mathcal{A} \cap \mathcal{B}$ *(the center of* $\mathcal{A}$*). Then* T *is diagonalized by the decomposition* (4.15) *and acts by a scalar* $\hat{T}(\lambda) \in \mathbb{C}$ *on* $E^\lambda \otimes F^\lambda$. *Conversely, given any complex-valued function* f *on* $\mathrm{Spec}(\rho)$, *there exists* $T \in \mathcal{A} \cap \mathcal{B}$ *such that* $\hat{T}(\lambda) = f(\lambda)$.

Proof. Since L is the direct sum of $\mathcal{B}$-invariant irreducible subspaces by Theorem 4.2.1, it is a completely reducible $\mathcal{B}$-module by Proposition 4.1.11. We now prove the other assertions.

(1): Let $T \in \mathrm{End}(L)$ be the operator that acts by $I \otimes T_\lambda$ on the λ summand. Then $T \in \mathrm{Comm}(\mathcal{B})$, and hence $T \in \mathcal{A}$ by the double commutant theorem (Theorem 4.1.13).

(2): Each summand in (4.15) is invariant under T, and the action of T on the λ summand is by an operator of the form $R_\lambda \otimes I = I \otimes S_\lambda$ with $R_\lambda \in \mathrm{End}(E^\lambda)$ and $S_\lambda \in \mathrm{End}(F^\lambda)$. Such an operator must be a scalar multiple of the identity operator. The converse follows from (1). □

4.2.2 Products of Reductive Groups

We now apply the duality theorem to determine the regular representations of the product of two reductive linear algebraic groups H and K. Let $G = H \times K$ be the direct product linear algebraic group. Recall that $\mathcal{O}[G] \cong \mathcal{O}[H] \otimes \mathcal{O}[K]$ under the natural pointwise multiplication map. Let (σ, V) and (τ, W) be regular representations of H and K respectively. The *outer tensor product* is the representation $(\sigma \widehat{\otimes} \tau, V \otimes W)$ of $H \times K$, where

$$(\sigma\widehat{\otimes}\tau)(h,k) = \sigma(h)\otimes\tau(k) \quad \text{for } h\in H \text{ and } k\in K\,.$$

Notice that when $H = K$, the restriction of the outer tensor product $\sigma\widehat{\otimes}\tau$ to the diagonal subgroup $\{(h,h) : h\in H\}$ of $H\times H$ is the tensor product $\sigma\otimes\tau$.

Proposition 4.2.5. *Suppose (σ,V) and (τ,W) are irreducible. Then the outer tensor product $(\sigma\widehat{\otimes}\tau, V\otimes W)$ is an irreducible representation of $H\times K$, and every irreducible regular representation of $H\times K$ is of this form.*

Proof. We have $\mathrm{End}(V\otimes W) = \mathrm{End}(V)\otimes\mathrm{End}(W) = \mathrm{Span}\{\sigma(H)\otimes\tau(K)\}$ by Corollary 4.1.7. Hence if $0\neq u\in V\otimes W$, then $\mathrm{Span}\{(\sigma(H)\otimes\tau(K))u\} = V\otimes W$. This shows that $\rho\widehat{\otimes}\sigma$ is irreducible.

Conversely, given an irreducible regular representation (ρ,L) of $H\times K$, set $\tau(k) = \rho(1,k)$ for $k\in K$, and use Theorem 4.2.1 (with $\mathcal{R} = \mathrm{End}(L)$) to decompose L as a K-module:

$$L = \bigoplus_{\lambda\in\mathrm{Spec}(\tau)} E^\lambda\otimes F^\lambda\,. \tag{4.16}$$

Set $\sigma(h) = \rho(h,1)$ for $h\in H$. Then $\sigma(H)\subset\mathrm{End}_K(L)$, and thus H preserves decomposition (4.16) and acts on the λ summand by $h\mapsto\sigma_\lambda(h)\otimes I$ for some representation σ_λ. We claim that σ_λ is irreducible. To prove this, note that since $\mathrm{End}_K(L)$ acts irreducibly on E^λ by Theorem 4.2.1, we have

$$\mathrm{End}_K(L)\cong\bigoplus_{\lambda\in\mathrm{Spec}(\tau)}\mathrm{End}(E^\lambda)\otimes I\,. \tag{4.17}$$

But ρ is an irreducible representation, so $\mathrm{End}(L)$ is spanned by the transformations $\rho(h,k) = \sigma(h)\tau(k)$ with $h\in H$ and $k\in K$. Since K is reductive, there is a projection $T\mapsto T^\natural$ from $\mathrm{End}(L)$ onto $\mathrm{End}_K(L)$, and $\tau(k)^\natural$, for $k\in K$, acts by a scalar in each summand in (4.16) by Schur's lemma. Hence $\mathrm{End}(E^\lambda)$ is spanned by $\sigma_\lambda(H)$, proving that σ_λ is irreducible. Thus each summand in (4.16) is an irreducible module for $H\times K$, by the earlier argument. Since ρ is irreducible, there can be only one summand in (4.16). Hence $\rho = \sigma\widehat{\otimes}\tau$. □

Proposition 4.2.6. *If H and K are reductive linear algebraic groups, then $H\times K$ is reductive.*

Proof. Let ρ be a regular representation of $H\times K$. As in the proof of Proposition 4.2.5 we set $\tau(k) = \rho(1,k)$ for $k\in K$, and we use Theorem 4.2.1 (with $\mathcal{R} = \mathrm{End}(L)$) to obtain decomposition (4.16) of L as a K-module. Set $\sigma(h) = \rho(h,1)$ for $h\in H$. Then $\sigma(H)\subset\mathrm{End}_K(L)$, and thus we have a regular representation of H on E^λ for each $\lambda\in\mathrm{Spec}(\tau)$. Since H is reductive, these representations of H decompose as direct sums of irreducible representations. Using these decompositions in (4.16), we obtain a decomposition of L as a direct sum of representations of $H\times K$ that are irreducible by Proposition 4.2.5. □

4.2.3 Isotypic Decomposition of $\mathcal{O}[G]$

Let G be a reductive linear algebraic group. The group $G \times G$ is reductive by Proposition 4.2.6 and it acts on $\mathcal{O}[G]$ by left and right translations. Denote this representation by ρ:

$$\rho(y,z)f(x) = f(y^{-1}xz), \quad \text{for} \quad f \in \mathcal{O}[G] \text{ and } x,y,z \in G .$$

For each $\lambda \in \widehat{G}$ fix an irreducible representation (π^λ, F^λ) in the class λ. Denote by λ^* the class of the representation contragredient to λ. We choose the representations π^λ so that the vector space F^{λ^*} equals $(F^\lambda)^*$ and the operator $\pi^{\lambda^*}(g)$ equals $\pi^\lambda(g^{-1})^t$. We write $d_\lambda = \dim V^\lambda$ and call d_λ the *degree* of the representation. Note that $d_\lambda = d_{\lambda^*}$.

Theorem 4.2.7. *For $\lambda \in \widehat{G}$ define $\varphi_\lambda(v^* \otimes v)(g) = \langle v^*, \pi^\lambda(g)v\rangle$ for $g \in G$, $v^* \in V^{\lambda^*}$, and $v \in V$. Extend φ_λ to a linear map from $F^{\lambda^*} \otimes F^\lambda$ to $\mathcal{O}[G]$. Then the following hold:*

1. $\mathrm{Range}(\varphi_\lambda)$ *is independent of the choice of the model (π^λ, F^λ) and furnishes an irreducible regular representation of $G \times G$ isomorphic to $F^{\lambda^*} \widehat{\otimes} F^\lambda$.*
2. *Under the action of $G \times G$, the space $\mathcal{O}[G]$ decomposes as*

$$\mathcal{O}[G] = \bigoplus_{\lambda \in \widehat{G}} \varphi_\lambda\left(F^{\lambda^*} \otimes F^\lambda\right) . \tag{4.18}$$

Proof. Given $v \in F^\lambda$ and $v^* \in F^{\lambda^*}$, we set $f_{v^*,v} = \varphi_\lambda(v^* \otimes v)$. Then for $x,y,z \in G$ we have

$$f_{x \cdot v^*, y \cdot v}(z) = \langle \pi^{\lambda^*}(x)v^*, \pi^\lambda(z)\pi^\lambda(y)v\rangle = f_{v^*,v}(x^{-1}zy) .$$

This shows that φ_λ intertwines the action of $G \times G$. Since $F^{\lambda^*} \otimes F^\lambda$ is irreducible as a $G \times G$ module by Proposition 4.2.5, Schur's lemma implies that φ_λ must be injective. It is clear that the range of φ_λ depends only on the equivalence class of (π^λ, F^λ).

Let $\mathcal{O}[G]_{(\lambda)}$ be the λ-isotypic subspace relative to the right action R of G. The calculation above shows that $\mathrm{Range}(\varphi_\lambda) \subset \mathcal{O}[G]_{(\lambda)}$, so by Proposition 4.1.15 we need to show only the opposite inclusion. Let $W \subset \mathcal{O}[G]_{(\lambda)}$ be any irreducible subspace for the right action of G. We may then take W as the model for λ in the definition of the map φ_λ. Define $\delta \in W^*$ by $\langle \delta, w\rangle = w(1)$. Then

$$f_{\delta,w}(g) = \langle \delta, R(g)w\rangle = (R(g)w)(1) = w(g) .$$

Hence $\varphi_\lambda(\delta \otimes w) = w$, completing the proof. □

Corollary 4.2.8. *In the right-translation representation of G on $\mathcal{O}[G]$ every irreducible representation of G occurs with multiplicity equal to its dimension.*

Remark 4.2.9. The representations of $G \times G$ that occur in $\mathcal{O}[G]$ are the outer tensor products $\lambda^* \widehat{\otimes} \lambda$ for all $\lambda \in \widehat{G}$, and each representation occurs with multiplicity one.

Under the isomorphism $F^{\lambda *} \otimes F^{\lambda} \cong \mathrm{End}(F^{\lambda})$ that sends the tensor $v^* \otimes v$ to the rank-one operator $u \mapsto \langle v^*, u \rangle v$, the map φ_λ in Theorem 4.2.7 is given by $\varphi_\lambda(T)(g) = \mathrm{tr}(\pi^\lambda(g)T)$ for $T \in \mathrm{End}(F^\lambda)$. In this model for the isotypic components the element $(g, g') \in G \times G$ acts by $T \mapsto \pi^\lambda(g')T\pi^\lambda(g)^{-1}$ on the λ summand.

The *duality principle* in Theorem 4.2.1 asserts that the commutant of G explains the multiplicities in the primary decomposition. For example, the space $\mathcal{O}[G]$ is a direct sum of irreducible representations of G, relative to the right translation action, since G is reductive. Obtaining such a decomposition requires decomposing each isotypic component into irreducible subspaces. If G is not an algebraic torus, then it has irreducible representations of dimension greater than one, and the decomposition of the corresponding isotypic component is not unique. However, when we include the additional symmetries coming from the commuting left translation action by G, then each isotypic component becomes irreducible under the action of $G \times G$.

4.2.4 Schur–Weyl Duality

We now apply the duality theorem to obtain a result that will play a central role in our study of tensor and polynomial invariants for the classical groups. Let ρ be the defining representation of $\mathbf{GL}(n, \mathbb{C})$ on $\mathbb{C}^n$. For all integers $k \geq 0$ we can construct the representation $\rho_k = \rho^{\otimes k}$ on $\bigotimes^k \mathbb{C}^n$. Since

$$\rho_k(g)(v_1 \otimes \cdots \otimes v_k) = gv_1 \otimes \cdots \otimes gv_k$$

for $g \in \mathbf{GL}(n, \mathbb{C})$, we can permute the positions of the vectors in the tensor product without changing the action of G. Let $\mathfrak{S}_k$ be the group of permutations of $\{1, 2, \ldots, k\}$. We define a representation σ_k of $\mathfrak{S}_k$ on $\bigotimes^k \mathbb{C}^n$ by

$$\sigma_k(s)(v_1 \otimes \cdots \otimes v_k) = v_{s^{-1}(1)} \otimes \cdots \otimes v_{s^{-1}(k)}$$

for $s \in \mathfrak{S}_k$. Notice that $\sigma_k(s)$ moves the vector in the ith position in the tensor product to the position $s(i)$. It is clear that $\sigma_k(s)$ commutes with $\rho_k(g)$ for all $s \in \mathfrak{S}_k$ and $g \in \mathbf{GL}(n, \mathbb{C})$. Let $\mathcal{A} = \mathrm{Span}\,\rho_k(\mathbf{GL}(n, \mathbb{C}))$ and $\mathcal{B} = \mathrm{Span}\,\sigma_k(\mathfrak{S}_k)$. Then we have $\mathcal{A} \subset \mathrm{Comm}(\mathcal{B})$.

Theorem 4.2.10 (Schur). *One has* $\mathrm{Comm}(\mathcal{B}) = \mathcal{A}$ *and* $\mathrm{Comm}(\mathcal{A}) = \mathcal{B}$.

Proof. The representations σ_k and ρ_k are completely reducible (by Corollary 3.3.6 and Theorem 3.3.11). From the double commutant theorem (Theorem 4.1.13) it suffices to prove that $\mathrm{Comm}(\mathcal{B}) \subset \mathcal{A}$.

Let $\{e_1, \ldots, e_n\}$ be the standard basis for $\mathbb{C}^n$. For an ordered k-tuple $I = (i_1, \ldots, i_k)$ with $1 \leq i_j \leq n$, define $|I| = k$ and $e_I = e_{i_1} \otimes \cdots \otimes e_{i_k}$. The tensors $\{e_I\}$, with I ranging over the all such k-tuples, give a basis for $\bigotimes^k \mathbb{C}^n$. The group $\mathfrak{S}_k$ permutes this basis by the action $\sigma_k(s)e_I = e_{s \cdot I}$, where for $I = (i_1, \ldots, i_k)$ and $s \in \mathfrak{S}_k$ we define

$$s \cdot (i_1, \ldots, i_k) = (i_{s^{-1}(1)}, \ldots, i_{s^{-1}(k)}) .$$

Note that s changes the *positions* (1 to k) of the indices, not their values (1 to n), and we have $(st) \cdot I = s \cdot (t \cdot I)$ for $s, t \in \mathfrak{S}_k$.

Suppose $T \in \mathrm{End}(\bigotimes^k \mathbb{C}^n)$ has matrix $[a_{I,J}]$ relative to the basis $\{e_I\}$:

$$Te_J = \sum_I a_{I,J} e_I .$$

We have

$$T(\sigma_k(s)e_J) = T(e_{s \cdot J}) = \sum_I a_{I,s \cdot J} e_I$$

for $s \in \mathfrak{S}_k$, whereas

$$\sigma_k(s)(Te_J) = \sum_I a_{I,J} e_{s \cdot I} = \sum_I a_{s^{-1} \cdot I, J} e_I .$$

Thus $T \in \mathrm{Comm}(\mathcal{B})$ if and only if $a_{I,s \cdot J} = a_{s^{-1} \cdot I,J}$ for all multi-indices I, J and all $s \in \mathfrak{S}_k$. Replacing I by $s \cdot I$, we can write this condition as

$$a_{s \cdot I, s \cdot J} = a_{I,J} \quad \text{for all } I, J \text{ and all } s \in \mathfrak{S}_k . \tag{4.19}$$

Consider the nondegenerate bilinear form $(X,Y) = \mathrm{tr}(XY)$ on $\mathrm{End}\left(\bigotimes^k \mathbb{C}^n\right)$. We claim that the restriction of this form to $\mathrm{Comm}(\mathcal{B})$ is nondegenerate. Indeed, we have a projection $X \mapsto X^\natural$ of $\mathrm{End}\left(\bigotimes^k \mathbb{C}^n\right)$ onto $\mathrm{Comm}(\mathcal{B})$ given by averaging over $\mathfrak{S}_k$:

$$X^\natural = \frac{1}{k!} \sum_{s \in \mathfrak{S}_k} \sigma_k(s) X \sigma_k(s)^{-1} .$$

If $T \in \mathrm{Comm}(\mathcal{B})$ then

$$(X^\natural, T) = \frac{1}{k!} \sum_{s \in \mathfrak{S}_k} \mathrm{tr}(\sigma_k(s) X \sigma_k(s)^{-1} T) = (X, T) ,$$

since $\sigma_k(s)T = T\sigma_k(s)$. Thus $(\mathrm{Comm}(\mathcal{B}), T) = 0$ implies that $(X,T) = 0$ for all $X \in \mathrm{End}\left(\bigotimes^k \mathbb{C}^n\right)$, and so $T = 0$. Hence the trace form on $\mathrm{Comm}(\mathcal{B})$ is nondegenerate.

To prove that $\mathrm{Comm}(\mathcal{B}) = \mathcal{A}$, it thus suffices to show that if $T \in \mathrm{Comm}(\mathcal{B})$ is orthogonal to $\mathcal{A}$ then $T = 0$. Now if $g = [g_{ij}] \in \mathbf{GL}(n, \mathbb{C})$, then $\rho_k(g)$ has matrix $g_{I,J} = g_{i_1 j_1} \cdots g_{i_k j_k}$ relative to the basis $\{e_I\}$. Thus we assume that

$$(T, \rho_k(g)) = \sum_{I,J} a_{I,J} g_{j_1 i_1} \cdots g_{j_k i_k} = 0 \tag{4.20}$$

for all $g \in \mathbf{GL}(n, \mathbb{C})$, where $[a_{I,J}]$ is the matrix of T. Define a polynomial function f_T on $M_n(\mathbb{C})$ by

$$f_T(X) = \sum_{I,J} a_{I,J} x_{j_1 i_1} \cdots x_{j_k i_k}$$

for $X = [x_{ij}] \in M_n(\mathbb{C})$. From (4.20) we have $\det(X) f_T(X) = 0$ for all $X \in M_n(\mathbb{C})$. Hence f_T is identically zero, so for all $[x_{ij}] \in M_n(\mathbb{C})$ we have

$$\sum_{I,J} a_{I,J} x_{j_1 i_1} \cdots x_{j_k i_k} = 0 . \tag{4.21}$$

We now show that (4.19) and (4.21) imply that $a_{I,J} = 0$ for all I,J. We begin by grouping the terms in (4.21) according to distinct monomials in the matrix entries $\{x_{ij}\}$. Introduce the notation $x_{I,J} = x_{i_1 j_1} \cdots x_{i_k j_k}$, and view these monomials as polynomial functions on $M_n(\mathbb{C})$. Let Ξ be the set of all ordered pairs (I,J) of multi-indices with $|I| = |J| = k$. The group $\mathfrak{S}_k$ acts on Ξ by

$$s \cdot (I,J) = (s \cdot I, s \cdot J) .$$

From (4.19) we see that T commutes with $\mathfrak{S}_k$ if and only if the function $(I,J) \mapsto a_{I,J}$ is constant on the orbits of $\mathfrak{S}_k$ in Ξ.

The action of $\mathfrak{S}_k$ on Ξ defines an equivalence relation on Ξ, where $(I,J) \equiv (I',J')$ if $(I',J') = (s \cdot I, s \cdot J)$ for some $s \in \mathfrak{S}_k$. This gives a decomposition of Ξ into disjoint equivalence classes. Choose a set Γ of representatives for the equivalence classes. Then every monomial $x_{I,J}$ with $|I| = |J| = k$ can be written as x_γ for some $\gamma \in \Gamma$. Indeed, since the variables x_{ij} mutually commute, we have

$$x_\gamma = x_{s \cdot \gamma} \quad \text{for all } s \in \mathfrak{S}_k \text{ and } \gamma \in \Gamma .$$

Suppose $x_{I,J} = x_{I',J'}$. Then there must be an integer p such that $x_{i'_1 j'_1} = x_{i_p j_p}$. Call $p = 1'$. Similarly, there must be an integer $q \neq p$ such that $x_{i'_2 j'_2} = x_{i_q j_q}$. Call $q = 2'$. Continuing this way, we obtain a permutation

$$s : (1,2,\ldots,k) \to (1',2',\ldots,k')$$

such that $I = s \cdot I'$ and $J = s \cdot J'$. This proves that γ is uniquely determined by x_γ. For $\gamma \in \Gamma$ let $n_\gamma = |\mathfrak{S}_k \cdot \gamma|$ be the cardinality of the corresponding orbit.

Assume that the coefficients $a_{I,J}$ satisfy (4.19) and (4.21). Since $a_{I,J} = a_\gamma$ for all $(I,J) \in \mathfrak{S}_k \cdot \gamma$, it follows from (4.21) that

$$\sum_{\gamma \in \Gamma} n_\gamma a_\gamma x_\gamma = 0 .$$

Since the set of monomials $\{x_\gamma : \gamma \in \Gamma\}$ is linearly independent, this implies that $a_{I,J} = 0$ for all $(I,J) \in \Xi$. This proves that $T = 0$. Hence $\mathcal{A} = \mathrm{Comm}(\mathcal{B})$. □

From Theorems 4.2.1 and 4.2.10 we obtain a preliminary version of *Schur–Weyl duality:*

Corollary 4.2.11. *There are irreducible, mutually inequivalent $\mathfrak{S}_k$-modules E^λ and irreducible, mutually inequivalent $\mathbf{GL}(n,\mathbb{C})$-modules F^λ such that*

$$\bigotimes^k \mathbb{C}^n \cong \bigoplus_{\lambda \in \mathrm{Spec}(\rho_k)} E^\lambda \otimes F^\lambda$$

as a representation of $\mathfrak{S}_k \times \mathbf{GL}(n,\mathbb{C})$. *The representation* E^λ *uniquely determines* F^λ *and conversely.*

In Chapter 9 we shall determine the explicit form of the irreducible representations and the duality correspondence in Corollary 4.2.11.

4.2.5 Commuting Algebra and Highest-Weight Vectors

Let $\mathfrak{g}$ be a semisimple Lie algebra and let V be a finite-dimensional $\mathfrak{g}$-module. We shall apply the theorem of the highest weight to decompose the commuting algebra $\mathrm{End}_{\mathfrak{g}}(V)$ as a direct sum of full matrix algebras.

Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a choice of positive roots of $\mathfrak{h}$, and let $\mathfrak{g} = \mathfrak{n}^- + \mathfrak{h} + \mathfrak{n}^+$ be the associated triangular decomposition of $\mathfrak{g}$, as in Corollary 2.5.25. Set

$$V^{\mathfrak{n}^+} = \{v \in V : X \cdot v = 0 \quad \text{for all } X \in \mathfrak{n}^+\} .$$

Note that if $T \in \mathrm{End}_{\mathfrak{g}}(V)$ then it preserves $V^{\mathfrak{n}^+}$ and it preserves the weight space decomposition

$$V^{\mathfrak{n}^+} = \bigoplus_{\mu \in \mathcal{S}} V^{\mathfrak{n}^+}(\mu) .$$

Here $\mathcal{S} = \{\mu \in P_{++}(\mathfrak{g}) : V^{\mathfrak{n}^+}(\mu) \neq 0\}$. By Theorem 3.2.5 we can label the equivalence classes of irreducible $\mathfrak{g}$-modules by their highest weights. For each $\mu \in \mathcal{S}$ choose an irreducible representation (π^μ, V^μ) with highest weight μ.

Theorem 4.2.12. *The restriction map* $\varphi : T \mapsto T|_{V^{\mathfrak{n}^+}}$ *for* $T \in \mathrm{End}_{\mathfrak{g}}(V)$ *gives an algebra isomorphism*

$$\mathrm{End}_{\mathfrak{g}}(V) \cong \bigoplus_{\mu \in \mathcal{S}} \mathrm{End}(V^{\mathfrak{n}^+}(\mu)) . \tag{4.22}$$

For every $\mu \in \mathcal{S}$ *the space* $V^{\mathfrak{n}^+}(\mu)$ *is an irreducible module for* $\mathrm{End}_{\mathfrak{g}}(V)$. *Furthermore, distinct values of* μ *give inequivalent modules for* $\mathrm{End}_{\mathfrak{g}}(V)$. *Under the joint action of* $\mathfrak{g}$ *and* $\mathrm{End}_{\mathfrak{g}}(V)$ *the space* V *decomposes as*

$$V \cong \bigoplus_{\mu \in \mathcal{S}} V^\mu \otimes V^{\mathfrak{n}^+}(\mu) . \tag{4.23}$$

Proof. Since every finite-dimensional representation of $\mathfrak{g}$ is completely reducible by Theorem 3.3.12, we can apply Proposition 4.1.15 (viewing V as a $U(\mathfrak{g})$-module) to obtain the primary decomposition

$$V = \bigoplus_{\mu \in P_{++}(\mathfrak{g})} V_{(\mu)}, \qquad \mathrm{End}_{\mathfrak{g}}(V) \cong \bigoplus_{\mu \in P_{++}(\mathfrak{g})} \mathrm{End}_{\mathfrak{g}}(V_{(\mu)}) . \tag{4.24}$$

Here we write $V_{(\mu)}$ for the isotypic component of V of type V^μ. For each $V_{(\mu)} \neq 0$ we choose irreducible submodules $V_{\mu,i} \cong V^\mu$ for $i = 1, \ldots, d(\mu)$ such that

$$V_{(\mu)} = V_{\mu,1} \oplus \cdots \oplus V_{\mu,d(\mu)} , \tag{4.25}$$

where $d(\mu) = \mathrm{mult}_V(\pi^\mu)$. Let $v_{\mu,i} \in V_{\mu,i}$ be a highest-weight vector. Then (4.25) and Corollary 3.3.14 imply that

$$\mathrm{mult}_V(\pi^\mu) = \dim V^{\mathfrak{n}^+}(\mu) . \tag{4.26}$$

Hence the nonzero terms in (4.24) are those with $\mu \in \mathcal{S}$.

Let $T \in \mathrm{End}_{\mathfrak{g}}(V)$ and suppose $\varphi(T) = 0$. Then $Tv_{\mu,i} = 0$ for all μ and $i = 1, \ldots, d(\mu)$. If $v = x_1 \cdots x_p v_{\mu,i}$ with $x_i \in \mathfrak{g}$, then

$$Tv = x_1 \cdots x_p T v_{\mu,i} = 0 .$$

But $v_{\mu,i}$ is a cyclic vector for $V_{\mu,i}$ by Theorem 3.2.5, so $TV_{\mu,i} = 0$. Hence $TV_{(\mu)} = 0$ for all $\mu \in P_{++}(\mathfrak{g})$. Thus $T = 0$, which shows that φ is injective. We also have

$$\dim \mathrm{End}_{\mathfrak{g}}(V_{(\mu)}) = (\mathrm{mult}_V(\pi^\mu))^2 = \big(\dim V^{\mathfrak{n}^+}(\mu)\big)^2 = \dim\big(\mathrm{End}\, V^{\mathfrak{n}^+}(\mu)\big)$$

by (4.26). Since φ is injective, it follows that $\varphi(\mathrm{End}_{\mathfrak{g}}(V_{(\mu)})) = \mathrm{End}(V^{\mathfrak{n}^+}(\mu))$ for all $\mu \in P_{++}(\mathfrak{g})$. Hence by (4.24) we see that φ is also surjective. This proves (4.22). The other assertions of the theorem now follow from (4.22) and (4.25). □

4.2.6 Abstract Capelli Theorem

Let G be a reductive linear algebraic group, and let (ρ, L) be a locally regular representation of G with $\dim L$ countable. Recall that ρ is a locally completely reducible representation of $\mathcal{A}[G]$, and the irreducible $\mathcal{A}[G]$ submodules of L are irreducible regular representations of G.

There is a representation $d\rho$ of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ on L such that on every finite-dimensional G-submodule $V \subset L$ one has $d\rho|_V = d(\rho|_V)$. We extend $d\rho$ to a representation of the universal enveloping algebra $U(\mathfrak{g})$ on L (see Appendix C.2.1). Denote by $Z(\mathfrak{g})$ the *center* of the algebra $U(\mathfrak{g})$ (the elements T such that $TX = XT$ for all $X \in \mathfrak{g}$). Assume that $\mathcal{R} \subset \mathrm{End}(L)$ is a subalgebra that satisfies the conditions (i), (ii), (iii) in Section 4.2.1.

Theorem 4.2.13. *Suppose $\mathcal{R}^G \subset d\rho(U(\mathfrak{g}))$. Then $\mathcal{R}^G \subset d\rho(Z(\mathfrak{g}))$ and $\mathcal{R}^G$ is commutative. Furthermore, in the decomposition (4.15) the irreducible $\mathcal{R}^G$-modules E^λ are all one-dimensional. Hence L is multiplicity-free as a G-module.*

Proof. Let τ be the representation of G on $\mathcal{R}$ given by

$$\tau(g)r = \rho(g)r\rho(g^{-1}).$$

Then $\tau(g) \in \mathrm{Aut}(\mathcal{R})$ and the representation τ is locally regular, by conditions (ii) and (iii) of Section 4.2.1. Hence there is a representation $d\tau : \mathfrak{g} \longrightarrow \mathrm{End}(\mathcal{R})$ such that on every finite-dimensional G-submodule $\mathcal{W} \subset \mathcal{R}$ one has $d\tau|_{\mathcal{W}} = d(\tau|_{\mathcal{W}})$. We claim that

$$d\tau(X)T = [d\rho(X), T] \qquad \text{for } X \in \mathfrak{g} \text{ and } T \in \mathcal{R}. \tag{4.27}$$

Indeed, given $v \in L$ and $T \in \mathcal{R}$, there are finite-dimensional G-submodules $V_0 \subset V_1 \subset L$ and $\mathcal{W} \subset \mathcal{R}$ such that $v \in V$, $T \in \mathcal{W}$, and $TV_0 \in V_1$. Thus the functions

$$t \mapsto \rho(\exp tX)T\rho(\exp -tX)v \quad \text{and} \quad t \mapsto \rho(\exp tX)T\rho(\exp -tX)$$

are analytic from $\mathbb{C}$ to the finite-dimensional spaces V_1 and $\mathcal{W}$, respectively. By definition of the differential of a representation,

$$\begin{aligned}(d\tau(X)T)v &= \frac{d}{dt}\rho(\exp tX)T\rho(\exp -tX)v\Big|_{t=0} \\ &= \frac{d}{dt}\rho(\exp tX)Tv\Big|_{t=0} + T\frac{d}{dt}\rho(\exp -tX)v\Big|_{t=0} \\ &= [d\rho(X), T]v,\end{aligned}$$

proving (4.27).

Now suppose $T \in \mathcal{R}^G$. Then $\tau(g)T = T$ for all $g \in G$, so $d\tau(X)T = 0$ for all $X \in \mathfrak{g}$. Hence by (4.27) we have

$$d\rho(X)T = Td\rho(X) \quad \text{for all } X \in \mathfrak{g}. \tag{4.28}$$

By assumption, there exists $\widetilde{T} \in U(\mathfrak{g})$ such that $T = d\rho(\widetilde{T})$. One has $\widetilde{T} \in U_k(\mathfrak{g})$ for some integer k. Set $K = U_k(\mathfrak{g}) \cap \mathrm{Ker}(d\rho)$. Then $\mathrm{Ad}(G)K = K$. Since G is reductive and the adjoint representation of G on $U_k(\mathfrak{g})$ is regular, there is an $\mathrm{Ad}(G)$-invariant subspace $M \subset U_k(\mathfrak{g})$ such that

$$U_k(\mathfrak{g}) = K \oplus M.$$

Write $\widetilde{T} = T_0 + T_1$, where $T_0 \in K$ and $T_1 \in M$. Then $d\rho(\widetilde{T}) = d\rho(T_1) = T$. From (4.28) we have

$$d\rho(\mathrm{ad}(X)T_1) = [d\rho(X), d\rho(T_1)] = 0 \quad \text{for all } X \in \mathfrak{g}.$$

Hence $\mathrm{ad}(X)T_1 \in \mathrm{Ker}(d\rho)$ for all $X \in \mathfrak{g}$. But the subspace M is invariant under $\mathrm{ad}(\mathfrak{g})$, since it is invariant under G. Thus

$$\mathrm{ad}(X)T_1 \in \mathrm{Ker}(d\rho) \cap M = \{0\} \quad \text{for all } X \in \mathfrak{g}.$$

This proves that $T_1 \in Z(\mathfrak{g})$. The algebra $d\rho(Z(\mathfrak{g}))$ is commutative, since $d\rho$ is a homomorphism of associative algebras. Hence the subalgebra $\mathcal{R}^G$ is commutative.

To prove that the irreducible $\mathcal{R}^G$-module E^λ has dimension one, let $\mathcal{B} = \mathcal{R}^G|_{E^\lambda}$. Then $\mathcal{B} \subset \mathrm{End}_{\mathcal{B}}(E^\lambda)$, since $\mathcal{B}$ is commutative. Hence by Schur's lemma (Lemma 4.1.4), we have $\dim \mathcal{B} = 1$, and hence $\dim E^\lambda = 1$. Since E^λ uniquely determines λ, it follows that L is multiplicity-free as a G-module. □

4.2.7 Exercises

1. Let $\mathcal{A}$ be an associative algebra with 1 and let $L : \mathcal{A} \longrightarrow \mathrm{End}(\mathcal{A})$ be the *left multiplication representation* $L(a)x = ax$. Suppose $T \in \mathrm{End}(\mathcal{A})$ commutes with $L(\mathcal{A})$. Prove that there is an element $b \in \mathcal{A}$ such that $T(a) = ab$ for all $a \in \mathcal{A}$. (HINT: Consider the action of T on 1.)
2. Let G be a group. Suppose $T \in \mathrm{End}(\mathcal{A}[G])$ commutes with left translations by G. Show that there is a function $\varphi \in \mathcal{A}[G]$ such that $Tf = f * \varphi$ (convolution product) for all $f \in \mathcal{A}[G]$. (HINT: Use the previous exercise.)
3. Let G be a linear algebraic group and (ρ, V) a regular representation of G. Define a representation π of $G \times G$ on $\mathrm{End}(V)$ by $\pi(x,y)T = \rho(x)T\rho(y^{-1})$ for $T \in \mathrm{End}(V)$ and $x, y \in G$.
(a) Show that the space E^ρ of representative functions (see Section 1.5.1) is invariant under $G \times G$ (acting by left and right translations) and that the map $B \mapsto f_B$ from $\mathrm{End}(V)$ to E^ρ intertwines the actions π and $L \widehat{\otimes} R$ of $G \times G$.
(b) Suppose ρ is irreducible. Prove that the map $B \mapsto f_B$ from $\mathrm{End}(V)$ to $\mathcal{O}[G]$ is injective. (HINT: Use Corollary 4.1.7.)

4.3 Group Algebras of Finite Groups

In this section apply the general results of the chapter to the case of the group algebra of a finite group, and we obtain the representation-theoretic version of Fourier analysis.

4.3.1 Structure of Group Algebras

Let G be a finite group. Thus $\mathcal{A}[G]$ consists of all complex-valued functions on G. We denote by L and R the left and right translation representations of G on $\mathcal{A}[G]$:

$$L(g)f(x) = f(g^{-1}x), \quad R(g)f(x) = f(xg).$$

By Corollary 3.3.6 we know that G is a reductive group. Each irreducible representation is finite-dimensional and has a G-invariant positive definite Hermitian inner product (obtained by averaging any inner product over G). Thus we may take each model (π^λ, F^λ) to be unitary for $\lambda \in \widehat{G}$. The space F^{λ^*} can be taken as F^λ with

$$\pi^{\lambda^*}(g) = \overline{\pi^\lambda(g)}\,. \tag{4.29}$$

Here the bar denotes the complex conjugate of the matrix of $\pi^\lambda(g)$ relative to any orthonormal basis for F^λ. Equation (4.29) holds because the transpose inverse of a unitary matrix is the complex conjugate of the matrix.

From Theorem 4.2.7 and Remark 4.2.9 the vector space $\mathcal{A}[G]$ decomposes under $G \times G$ as

$$\mathcal{A}[G] \cong \bigoplus_{\lambda \in \widehat{G}} \mathrm{End}(F^\lambda)\,, \tag{4.30}$$

with $(g,h) \in G \times G$ acting on $T \in \mathrm{End}(F^\lambda)$ by $T \mapsto \pi^\lambda(g)T\pi^\lambda(h)^{-1}$. In particular, since $\dim \mathcal{A}[G] = |G|$ and $\dim \mathrm{End}\,(F^\lambda) = (d_\lambda)^2$, the isomorphism (4.30) implies that

$$|G| = \sum_{\lambda \in \widehat{G}} (d_\lambda)^2\,. \tag{4.31}$$

We recall that $\mathcal{A}[G]$ is an associative algebra relative to the convolution product, with identity element δ_1. It has a conjugate-linear antiautomorphism $f \mapsto f^*$ given by

$$f^*(g) = \overline{f(g^{-1})}$$

(the conjugate-linear extension of the inversion map on G to $\mathcal{A}[G]$). If we view the right side of (4.30) as block diagonal matrices (one block for each $\lambda \in \widehat{G}$ and an element of $\mathrm{End}\,F^\lambda$ in the block indexed by λ), then these matrices also form an associative algebra under matrix multiplication. For $T \in \mathrm{End}\,F^\lambda$ let T^* denote the adjoint operator relative to the G-invariant inner product on F^λ:

$$(Tu, v) = (u, T^*v) \quad \text{for } u, v \in F^\lambda\,.$$

We define a conjugate-linear antiautomorphism of the algebra $\bigoplus_{\lambda \in \widehat{G}} \mathrm{End}(F^\lambda)$ by using the map $T \mapsto T^*$ on each summand.

We will now define an explicit isomorphism between these two algebras. Given $f \in \mathcal{A}[G]$ and $\lambda \in \widehat{G}$, we define an operator $\mathcal{F}f(\lambda)$ on F^λ by

$$\mathcal{F}f(\lambda) = \sum_{x \in G} f(x)\pi^\lambda(x)\,.$$

In particular, when f is the function δ_g with $g \in G$, then $\mathcal{F}\delta_g(\lambda) = \pi^\lambda(g)$. Hence the map $f \mapsto \mathcal{F}f(\lambda)$ is the canonical extension of the representation π^λ of G to a representation of $\mathcal{A}[G]$. We define the *Fourier transform* $\mathcal{F}f$ of f to be the element of the algebra $\bigoplus_{\lambda \in \widehat{G}} \mathrm{End}(F^\lambda)$ with λ component $\mathcal{F}f(\lambda)$.

Theorem 4.3.1. *The Fourier transform*

$$\mathcal{F} : A[G] \longrightarrow \bigoplus_{\lambda \in \widehat{G}} \mathrm{End}(F^\lambda)$$

is an algebra isomorphism that preserves the $*$ operation on each algebra. Furthermore, for $f \in A[G]$ and $g \in G$ one has

$$\mathcal{F}(L(g)f)(\lambda) = \pi^\lambda(g)\mathcal{F}f(\lambda)\,, \qquad \mathcal{F}(R(g)f)(\lambda) = \mathcal{F}f(\lambda)\pi^\lambda(g^{-1})\,. \qquad (4.32)$$

Proof. Since $\mathcal{F}(\delta_{g_1g_2}) = \pi^\lambda(g_1g_2) = \pi^\lambda(g_1)\pi^\lambda(g_2) = \mathcal{F}(\delta_{g_1})\mathcal{F}(\delta_{g_2})$, the map $\mathcal{F}$ transforms convolution of functions on G into multiplication of operators on each space F^λ:

$$\mathcal{F}(f_1 * f_2)(\lambda) = \mathcal{F}f_1(\lambda)\mathcal{F}f_2(\lambda)$$

for $f_1, f_2 \in A[G]$ and $\lambda \in \widehat{G}$. Also, $\mathcal{F}\delta_1(\lambda) = I_{F^\lambda}$. This shows that $\mathcal{F}$ is an algebra homomorphism. Hence equations (4.32) follow from $L(g)f = \delta_g * f$ and $R(g)f = f * \delta_{g^{-1}}$. The $*$ operation is preserved by $\mathcal{F}$, since $(\delta_g)^* = \delta_{g^{-1}}$ and $(\pi^\lambda(g))^* = \pi^\lambda(g^{-1})$. From Corollary 4.2.4 (1) we see that $\mathcal{F}$ is surjective. Then (4.31) shows that it is bijective. □

4.3.2 Schur Orthogonality Relations

We begin with a variant of Schur's lemma. Let G be a group and let U and V be finite-dimensional G-modules.

Lemma 4.3.2. *Suppose C is a G-invariant bilinear form on $U \times V$. Then $C = 0$ if U is not equivalent to V^* as a G-module. If $U = V^*$ there is a constant κ such that $C(u,v) = \kappa\langle u,v\rangle$, where $\langle u,v\rangle$ denotes the canonical bilinear pairing of V^* and V.*

Proof. We can write C as $C(u,v) = \langle Tu, v\rangle$, where $T \in \mathrm{Hom}(U,V^*)$. Since the form C and the canonical bilinear pairing of V^* and V are both G invariant, we have

$$\langle g^{-1}Tgu, v\rangle = \langle Tgu, gv\rangle = \langle Tu, v\rangle$$

for all $u \in U$, $v \in V$, and $g \in G$. Hence $g^{-1}Tg = T$, and so $T \in \mathrm{Hom}_G(U,V^*)$. The conclusion now follows from Lemma 4.1.4. □

Let $\lambda \in \widehat{G}$. For $A \in \mathrm{End}(F^\lambda)$ we define the *representative function* f^λ_A on G by $f^\lambda_A(g) = \mathrm{tr}(\pi^\lambda(g)A)$ for $g \in G$, as in Section 1.5.1.

Lemma 4.3.3 (Schur Orthogonality Relations). *Suppose G is a finite group and $\lambda, \mu \in \widehat{G}$. Let $A \in \mathrm{End}(F^\lambda)$ and $B \in \mathrm{End}(F^\mu)$. Then*

$$\frac{1}{|G|}\sum_{g\in G} f^\lambda_A(g) f^\mu_B(g) = \begin{cases} (1/d_\lambda)\,\mathrm{tr}(AB^t) & \text{if } \mu = \lambda^*\,, \\ 0 & \text{otherwise.} \end{cases} \qquad (4.33)$$

Proof. Define a bilinear form C on $\mathrm{End}(F^\lambda) \times \mathrm{End}(F^\mu)$ by

$$C(A,B) = \frac{1}{|G|} \sum_{g \in G} f_A^\lambda(g) f_B^\mu(g) . \tag{4.34}$$

We have $f_A^\lambda(xgy) = f^\lambda_{\pi^\lambda(y)A\pi^\lambda(x)}(g)$ for $x,y \in G$, with an analogous transformation law for f_B^μ. Replacing g by xgy in (4.34), we see that

$$C(\pi^\lambda(y)A\pi^\lambda(x),\ \pi^\mu(y)B\pi^\mu(x)) = C(A,B) . \tag{4.35}$$

Thus C is invariant under $G \times G$. Since $\mathrm{End}(F^\lambda)$ is an irreducible module for $G \times G$ (isomorphic to the outer tensor product module $F^\lambda \widehat{\otimes} F^{\lambda^*}$), Lemma 4.3.2 implies that $C = 0$ if $\mu \neq \lambda^*$.

Suppose now $\mu = \lambda^*$ and write $\pi = \pi^\lambda$, $V = F^\lambda$, $\pi^* = \pi^{\lambda^*}$, and $V^* = F^{\lambda^*}$. The bilinear form $\langle A,B \rangle = \mathrm{tr}_V(AB^t)$, for $A \in \mathrm{End}(V)$ and $B \in \mathrm{End}(V^*)$, is G-invariant and nondegenerate, so by Lemma 4.3.2 there is a constant κ such that $C(A,B) = \kappa\,\mathrm{tr}_V(AB^t)$. To determine κ, we recall that for $A \in \mathrm{End}(V)$ and $B \in \mathrm{End}(V^*)$ we have $\mathrm{tr}_V(A)\,\mathrm{tr}_{V^*}(B) = \mathrm{tr}_{V\otimes V^*}(A \otimes B)$. Thus

$$f_A^\lambda(g) f_B^\mu(g) = \mathrm{tr}_{V\otimes V^*}(\pi(g)A \otimes \pi^*(g)B) .$$

Now take $A = I_V$ and $B = I_{V^*}$. Then $\mathrm{tr}_V(AB^t) = d_\lambda$, and hence $\kappa d_\lambda = \mathrm{tr}_{V\otimes V^*}(P)$, where

$$P = \frac{1}{|G|} \sum_{g \in G} \pi(g) \otimes \pi^*(g)$$

is the projection onto the G-invariant subspace of $V \otimes V^*$. But by Lemma 4.3.2 this subspace has dimension one. Hence $\mathrm{tr}_{V\otimes V^*}(P) = 1$, proving that $\kappa = 1/d_\lambda$. □

4.3.3 Fourier Inversion Formula

With the aid of the Schur orthogonality relations we can now find an explicit inverse to the Fourier transform $\mathcal{F}$ on $A[G]$.

Theorem 4.3.4 (Fourier Inversion Formula). *Suppose G is a finite group. Let $F = \{F(\lambda)\}_{\lambda \in \widehat{G}}$ be in $\mathcal{F}A[G]$. Define a function $f \in A[G]$ by*

$$f(g) = \frac{1}{|G|} \sum_{\lambda \in \widehat{G}} d_\lambda\, \mathrm{tr}\left(\pi^\lambda(g) F(\lambda^*)^t\right) . \tag{4.36}$$

Then $\mathcal{F}f(\lambda) = F(\lambda)$ for all $\lambda \in \widehat{G}$.

Proof. The operator $\mathcal{F}f(\lambda)$ is uniquely determined by $\mathrm{tr}(\mathcal{F}f(\lambda)A)$, with A varying over $\mathrm{End}(V^\lambda)$. Replacing each representation by its dual, we write the formula for f as

$$f(g) = \frac{1}{|G|} \sum_{\mu \in \widehat{G}} d_\mu \operatorname{tr}\left(\pi^{\mu^*}(g) F(\mu)^t\right).$$

Then we calculate

$$\begin{aligned}\operatorname{tr}(\mathcal{F}f(\lambda)A) &= \frac{1}{|G|} \sum_{g \in G} \sum_{\mu \in \widehat{G}} d_\mu \operatorname{tr}\left(\pi^\lambda(g)A\right) \operatorname{tr}\left(\pi^{\mu^*}(g)F(\mu)^t\right) \\ &= \sum_{\mu \in \widehat{G}} d_\mu \Big\{ \frac{1}{|G|} \sum_{g \in G} f_A(g)\, f_{F(\mu)^t}(g) \Big\}.\end{aligned}$$

Applying the Schur orthogonality relations (4.33), we find that the terms with $\mu \neq \lambda$ vanish and $\operatorname{tr}(\mathcal{F}f(\lambda)A) = \operatorname{tr}(AF(\lambda))$. Since this holds for all A, we conclude that $\mathcal{F}f(\lambda) = F(\lambda)$. □

Corollary 4.3.5 (Plancherel Formula). *Let* $\varphi, \psi \in \mathcal{A}[G]$. *Then*

$$\sum_{g \in G} \varphi(g)\overline{\psi(g)} = \frac{1}{|G|} \sum_{\lambda \in \widehat{G}} d_\lambda \operatorname{tr}\left(\mathcal{F}\varphi(\lambda)\mathcal{F}\psi(\lambda)^*\right). \tag{4.37}$$

Proof. Let $f = \varphi * (\psi)^*$. Then

$$f(1) = \sum_{g \in G} \varphi(g)\overline{\psi(g)}.$$

We can also express $f(1)$ by the Fourier inversion formula evaluated at $g = 1$:

$$f(1) = \frac{1}{|G|} \sum_{\lambda \in \widehat{G}} d_\lambda \operatorname{tr}(\mathcal{F}f(\lambda^*)^t) = \frac{1}{|G|} \sum_{\lambda \in \widehat{G}} d_\lambda \operatorname{tr}(\mathcal{F}f(\lambda)).$$

Since $\mathcal{F}f(\lambda) = \mathcal{F}\varphi(\lambda)\mathcal{F}\psi(\lambda)^*$, we obtain (4.37). □

Remark 4.3.6. If we use the *normalized* Fourier transform $\Phi(\lambda) = (1/|G|)\mathcal{F}\phi(\lambda)$, then (4.37) becomes

$$\frac{1}{|G|} \sum_{g \in G} \varphi(g)\overline{\psi(g)} = \sum_{\lambda \in \widehat{G}} d_\lambda \operatorname{tr}\left(\Phi(\lambda)\Psi(\lambda)^*\right). \tag{4.38}$$

The left side of (4.38) is a positive definite Hermitian inner product on $\mathcal{A}[G]$ that is invariant under the operators $L(g)$ and $R(g)$ for $g \in G$, normalized so that the constant function $\varphi(g) = 1$ has norm 1. The Plancherel formula expresses this inner product in terms of the inner products on $\operatorname{End}(E^\lambda)$ given by $\operatorname{tr}(ST^*)$; these inner products are invariant under left and right multiplication by the unitary operators $\pi^\lambda(g)$ for $g \in G$. In this form the Plancherel formula applies to every compact topological group, with $\mathcal{A}(G)$ replaced by $L^2(G)$ and summation over G replaced by integration relative to the normalized invariant measure.

4.3.4 The Algebra of Central Functions

We continue our investigation of the group algebra of a finite group G. Let $\mathcal{A}[G]^G$ be the *center* of $\mathcal{A}[G]$: by definition, $f \in \mathcal{A}[G]^G$ if and only if

$$f * \varphi = \varphi * f \quad \text{for all } \varphi \in \mathcal{A}[G] . \tag{4.39}$$

We call such a function f a *central function* on G. The space of central functions on G is a commutative algebra (under convolution multiplication). In this section we shall write down two different bases for the space $\mathcal{A}[G]^G$ and use the Fourier transform on G to study the relation between them.

We first observe that in (4.39) it suffices to take $\varphi = \delta_x$, with x ranging over G, since these functions span $\mathcal{A}[G]$. Thus f is central if and only if $f(yx) = f(xy)$ for all $x, y \in G$. Replacing y by yx^{-1}, we can express this condition as

$$f(xyx^{-1}) = f(y) \quad \text{for } x, y \in G .$$

Thus we can also describe $\mathcal{A}[G]^G$ as the space of functions f on G that are constant on the conjugacy classes of G. From this observation we obtain the following basis for $\mathcal{A}[G]^G$:

Proposition 4.3.7. *Let* $\mathrm{Conj}(G)$ *be the set of conjugacy classes in* G. *For each* $C \in \mathrm{Conj}(G)$ *let* φ_C *be the characteristic function of* C. *Then the set* $\{\varphi_C\}_{C \in \mathrm{Conj}(G)}$ *is a basis for* $\mathcal{A}[G]^G$, *and every function* $f \in \mathcal{A}[G]^G$ *has the expansion*

$$f = \sum_{C \in \mathrm{Conj}(G)} f(C)\varphi_C$$

In particular,

$$\dim \mathcal{A}[G]^G = |\mathrm{Conj}(G)| . \tag{4.40}$$

We denote the character of a finite-dimensional representation ρ by χ_ρ, viewed as a function on G: $\chi_\rho(g) = \mathrm{tr}(\rho(g))$. Characters are central functions because

$$\mathrm{tr}(\rho(xy)) = \mathrm{tr}(\rho(x)\rho(y)) = \mathrm{tr}(\rho(y)\rho(x)) .$$

We note that

$$\chi_\rho(g^{-1}) = \overline{\chi_\rho(g)} , \tag{4.41}$$

where the bar denotes complex conjugate. Indeed, since $\rho(g)$ can be taken as a unitary matrix relative to a suitable basis, the eigenvalues of $\rho(g)$ have absolute value 1. Hence the eigenvalues of $\rho(g^{-1})$ are the complex conjugates of those of $\rho(g)$, and the trace is the sum of these eigenvalues. We write χ_λ for the character of the irreducible representation π^λ.

We have another representation of $\mathcal{A}[G]^G$ obtained from the Fourier transform. We know that the map $\mathcal{F}$ is an algebra isomorphism from $\mathcal{A}[G]$ (with convolution multiplication) to

$$\mathcal{F}\mathcal{A}[G] = \bigoplus_{\lambda \in \widehat{G}} \mathrm{End}(F^\lambda)$$

by Theorem 4.3.1. Since the center of each ideal $\mathrm{End}(F^\lambda)$ in $\mathcal{F}\mathcal{A}[G]$ consists of scalar multiples of the identity operator, we conclude that f is a central function on G if and only if

$$\mathcal{F}f(\lambda) = c_\lambda I_{F^\lambda} \quad \text{for all } \lambda \in \widehat{G}\,, \tag{4.42}$$

where $c_\lambda \in \mathbb{C}$. For each $\lambda \in \widehat{G}$ define $E_\lambda \in \mathcal{F}\mathcal{A}[G]$ to be the identity operator on F^λ and zero on F^μ for $\mu \neq \lambda$. The set of operator-valued functions $\{E_\lambda\}_{\lambda \in \widehat{G}}$ is obviously linearly independent, and from (4.42) we see that it is a basis for $\mathcal{F}\mathcal{A}[G]^G$.

Proposition 4.3.8. *The Fourier transform of $f \in \mathcal{A}[G]^G$ has the expansion*

$$\mathcal{F}f = \sum_{\lambda \in \widehat{G}} \mathcal{F}f(\lambda) E_\lambda\,. \tag{4.43}$$

In particular, $\dim \mathcal{A}[G]^G = |\widehat{G}|$, *and hence*

$$|\widehat{G}| = |\mathrm{Conj}(G)|\,. \tag{4.44}$$

Example

Suppose $G = \mathfrak{S}_n$, the symmetric group on n letters. Every $g \in G$ can be written uniquely as a product of disjoint cyclic permutations. For example, $(123)(45)$ is the permutation $1 \to 2, 2 \to 3, 3 \to 1, 4 \to 5, 5 \to 4$ in $\mathfrak{S}_5$. Furthermore, g is conjugate to g' if and only if the number of cycles of length $1, 2, \ldots, n$ is the same for g and g'. Thus each conjugacy class C in G corresponds to a *partition* of the integer n as the sum of positive integers:

$$n = k_1 + k_2 + \cdots + k_d\,,$$

with $k_1 \geq k_2 \geq \cdots \geq k_d > 0$. The class C consists of all elements with cycle lengths $k_1, k_2, \ldots, k_d$. From (4.44) it follows that $\mathfrak{S}_n$ has $p(n)$ inequivalent irreducible representations, where $p(n)$ is the number of partitions of n. □

We return to a general finite group G. Under the inverse Fourier transform, the operator E_λ corresponds to convolution by a central function e_λ on G. To determine e_λ, we apply the Fourier inversion formula (4.36):

$$e_\lambda(g) = \mathcal{F}^{-1}E_\lambda(g) = \frac{d_\lambda}{|G|}\chi_\lambda(g^{-1})\,. \tag{4.45}$$

Since $\mathcal{F}^{-1}$ is an algebra isomorphism, the family of functions $\{e_\lambda : \lambda \in \widehat{G}\}$ gives a resolution of the identity for the algebra $\mathcal{A}[G]$:

$$e_\lambda * e_\mu = \begin{cases} e_\lambda & \text{for } \lambda = \mu\,, \\ 0 & \text{otherwise}\,, \end{cases} \quad \text{and} \quad \sum_{\lambda \in \widehat{G}} e_\lambda = \delta_1\,. \tag{4.46}$$

Since $E_\lambda = \mathcal{F}e_\lambda$ and $\chi_\lambda(g^{-1}) = \chi_{\lambda^*}(g)$, we find from (4.45) that

$$\mathcal{F}\chi_{\lambda^*}(\mu) = \begin{cases} (|G|/d_\lambda)I_{F^\lambda} & \text{if } \mu = \lambda\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{4.47}$$

Thus the irreducible characters have Fourier transforms that vanish except on a single irreducible representation. Furthermore, from Proposition 4.3.8 we see that the irreducible characters give a basis for $\mathcal{A}[G]^G$. The explicit form of the expansion of a central function in terms of irreducible characters is as follows:

Theorem 4.3.9. *Let $\varphi, \psi \in \mathcal{A}[G]^G$ and $g \in G$. Then*

$$\varphi(g) = \sum_{\lambda\in\widehat{G}} \hat{\varphi}(\lambda)\chi_\lambda(g)\,, \text{ where } \hat{\varphi}(\lambda) = \frac{1}{|G|}\sum_{g\in G}\varphi(g)\overline{\chi_\lambda(g)}\,, \text{ and} \tag{4.48}$$

$$\frac{1}{|G|}\sum_{g\in G}\varphi(g)\overline{\psi(g)} = \sum_{\lambda\in\widehat{G}}\hat{\varphi}(\lambda)\overline{\hat{\psi}(\lambda)}\,. \tag{4.49}$$

Proof. Define a positive definite inner product on $\mathcal{A}[G]$ by

$$\langle\varphi \mid \psi\rangle = \frac{1}{|G|}\sum_{g\in G}\varphi(g)\overline{\psi(g)}\,.$$

Let $\lambda, \mu \in \widehat{G}$. Then $\chi_\lambda(g) = f_A^\lambda(g)$ and $\overline{\chi_\mu(g)} = f_B^{\mu^*}(g)$, where A is the identity operator on F^λ and B is the identity operator on F^{μ^*}. Hence the Schur orthogonality relations imply that

$$\langle\chi_\mu \mid \chi_\lambda\rangle = \begin{cases} 1 & \text{if } \mu = \lambda\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Thus $\{\chi_\lambda\}_{\lambda\in\widehat{G}}$ is an orthonormal basis for $\mathcal{A}[G]^G$, relative to the inner product $\langle\cdot|\cdot\rangle$. This implies formulas (4.48) and (4.49). □

Corollary 4.3.10 (Dual Orthogonality Relations). *Suppose C_1 and C_2 are conjugacy classes in G. Then*

$$\sum_{\lambda\in\widehat{G}}\chi_\lambda(C_1)\overline{\chi_\lambda(C_2)} = \begin{cases} |G|/|C_1| & \text{if } C_1 = C_2\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{4.50}$$

Proof. Let $C \subset G$ be a conjugacy class. Then

$$|G|\widehat{\varphi_C}(\lambda) = |C|\chi_{\lambda^*}(C)\,. \tag{4.51}$$

Taking $C = C_1$ and $C = C_2$ in (4.51) and then using (4.49), we obtain (4.50). □

Corollary 4.3.11. *Suppose (ρ, V) is any finite-dimensional representation of G. For $\lambda \in \widehat{G}$ let $m_\rho(\lambda)$ be the multiplicity of λ in ρ. Then $m_\rho(\lambda) = \langle\chi_\rho \mid \chi_\lambda\rangle$ and*

$$\langle\chi_\rho \mid \chi_\rho\rangle = \sum_{\lambda\in\widehat{G}} m_\rho(\lambda)^2\,.$$

In particular, $\langle \chi_\rho \mid \chi_\lambda \rangle$ *is a positive integer, and* ρ *is irreducible if and only if* $\langle \chi_\rho \mid \chi_\rho \rangle = 1$. *The operator*

$$P_\lambda = \frac{d_\lambda}{|G|} \sum_{g \in G} \overline{\chi_\lambda(g)}\, \rho(g) \tag{4.52}$$

is the projection onto the λ*-isotypic component of* V.

Proof. We have

$$\chi_\rho = \sum_{\lambda \in \widehat{G}} m_\rho(\lambda) \chi_\lambda ,$$

so the result on multiplicities follows from (4.48) and (4.49).

By Corollary 4.2.4 (2) there exists $f \in \mathcal{A}[G]^G$ such that $\rho(f) = P_\lambda$. To show that

$$f(g) = \frac{d_\lambda}{|G|} \overline{\chi_\lambda(g)} \quad \text{for } g \in G ,$$

it suffices (by complete reducibility of ρ) to show that $P_\lambda = \rho(f)$ when $\rho = \pi^\mu$ for some $\mu \in \widehat{G}$. In this case $\rho(f) = \delta_{\lambda\mu} I_{F^\mu}$ by (4.47), and the same formula holds for P_λ by definition. □

Finding an explicit formula for χ_λ or for $\mathcal{F}\varphi_C$ is a difficult problem whenever G is a noncommutative finite group. We shall solve this problem for the symmetric group in Chapter 9 by relating the representations of the symmetric group to representations of the general linear group.

Remark 4.3.12. The sets of functions $\{\chi_\lambda : \lambda \in \widehat{G}\}$ and $\{\varphi_C : C \in \mathrm{Conj}(G)\}$ on G have the same cardinality. However, there is no other simple relationship between them. This is a representation-theoretic version of the *uncertainty principle*: The function φ_C is supported on a single conjugacy class. If $C \neq \{1\}$ this forces $\mathcal{F}\varphi_C$ to be nonzero on at least two irreducible representations. (Let $\varphi = \varphi_C$ and $\psi = \delta_1$; then the left side of (4.49) is zero, while the right side is $\sum_\lambda d_\lambda \widehat{\varphi_C}(\lambda)$.) In the other direction, $\mathcal{F}\chi_\lambda$ is supported on the single irreducible representation λ. If λ is not the trivial representation, this forces χ_λ to be nonzero on at least two nontrivial conjugacy classes. (Since the trivial representation has character 1, the orthogonality of characters yields $\sum_{C \in \mathrm{Conj}(G)} |C| \chi_\lambda(C) = 0$.)

4.3.5 Exercises

1. Let $n > 1$ be an integer, and let $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ be the additive group of integers mod n.
   (a) Let $\mathrm{e}(k) = \mathrm{e}^{2\pi i k/n}$ for $k \in \mathbb{Z}_n$. Show that the characters of $\mathbb{Z}_n$ are the functions $\chi_q(k) = \mathrm{e}(kq)$ for $q = 0, 1, \ldots, n-1$.
   (b) For $f \in \mathcal{A}[\mathbb{Z}_n]$, define $\hat{f} \in \mathcal{A}[\mathbb{Z}_n]$ by $\hat{f}(q) = (1/n)\sum_{k=0}^{n-1} f(k)\mathrm{e}(-kq)$. Show that $f(k) = \sum_{q=0}^{n-1} \hat{f}(q)\mathrm{e}(kq)$, and that

$$\frac{1}{n}\sum_{k=0}^{n-1} |f(k)|^2 = \sum_{q=0}^{n-1} |\hat{f}(q)|^2 .$$

2. Let $\mathbb{F}$ be a finite field of characteristic p (so $\mathbb{F}$ has $q = p^n$ elements for some integer n). This exercise and several that follow apply Fourier analysis to the additive group of $\mathbb{F}$. This requires no detailed knowledge of the structure of $\mathbb{F}$ when p does not divide n; for the case that p divides n you will need to know more about finite fields to verify part (a) of the exercise. Let $\mathbb{S}^1$ be the multiplicative group of complex numbers of absolute value 1. Let $\chi : \mathbb{F} \to \mathbb{S}^1$ be such that $\chi(x+y) = \chi(x)\chi(y)$ and $\chi(0) = 1$ (i.e., χ is an *additive character* of $\mathbb{F}$). The smallest subfield of $\mathbb{F}$, call it $\mathbb{K}$, is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Define $\mathrm{e}(k) = \mathrm{e}^{2\pi i k/p}$ for $k \in \mathbb{Z}/p\mathbb{Z}$. This defines an additive character of $\mathbb{Z}/p\mathbb{Z}$ and hence a character of $\mathbb{K}$. $\mathbb{F}$ is a finite-dimensional vector space over $\mathbb{K}$. If $a \in \mathbb{F}$ define a linear transformation $L_a : \mathbb{F} \to \mathbb{F}$ by $L_a x = ax$. Set $\chi_1(a) = \mathrm{e}(\mathrm{tr}(L_a))$.
(a) We say that an additive character χ is *nontrivial* if $\chi(x) \neq 1$ for some $x \in \mathbb{F}$. Let u be a nonzero element of $\mathbb{F}$ and define $\eta(x) = \chi_1(ux)$. Show that η is a nontrivial additive character of $\mathbb{F}$.
(b) Show that if η is an additive character of $\mathbb{F}$ then there exists a unique $u \in \mathbb{F}$ such that $\eta(x) = \chi_1(ux)$ for all $x \in \mathbb{F}$.
(c) Show that if χ is any nontrivial additive character of $\mathbb{F}$ and if η is an additive character of $\mathbb{F}$ then there exists a unique $u \in \mathbb{F}$ such that $\eta(x) = \chi(ux)$ for all $x \in \mathbb{F}$.
3. Let $\mathbb{F}$ be a finite field. Fix a nontrivial additive character χ of $\mathbb{F}$. Show that the Fourier transform on $\mathcal{A}[\mathbb{F}]$ (relative to the additive group structure of $\mathbb{F}$) can be expressed as follows: For $f \in \mathcal{A}[\mathbb{F}]$, define $\hat{f} \in \mathcal{A}[\mathbb{F}]$ by

$$\hat{f}(\xi) = \frac{1}{|\mathbb{F}|}\sum_{x\in\mathbb{F}} f(y)\chi(-x\xi)$$

for $\xi \in \mathbb{F}$. Then the Fourier inversion formula becomes

$$f(x) = \sum_{\xi\in\mathbb{F}} \hat{f}(\xi)\chi(x\xi) ,$$

and one has $(1/|\mathbb{F}|)\sum_{x\in\mathbb{F}} |f(x)|^2 = \sum_{\xi\in\mathbb{F}} |\hat{f}(\xi)|^2$.

4.4 Representations of Finite Groups

Constructing irreducible representations of finite nonabelian groups is a difficult problem. As a preliminary step, we construct in this section a more accessible class of representations, the *induced representations*, and calculate their characters.