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Characters of Finite Coxeter Groups and Iwahori–Hecke Algebras

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Representation theory of symmetric algebras

In the previous chapters, we have studied the irreducible characters of finite Coxeter groups. For this purpose, we have only required some basic knowledge about the character theory of finite groups, as can be found, for example, in [Isaacs 1976]. In later chapters, we will want to study in a similar way the irreducible representations and characters of Iwahori–Hecke algebras. For example, we would like to have analogous versions of the usual orthogonality relations for the irreducible characters of a finite group. The purpose of this chapter is to lay the foundations for such a more general theory.

The common feature of group algebras of finite groups and Iwahori–Hecke algebras associated with finite Coxeter groups is that they are both examples of *symmetric* algebras. An (associative) algebra H is called symmetric if it carries a symmetric non-degenerate bilinear form $(\, , \,)$ such that $(ab, c) = (a, bc)$ for all $a, b, c \in H$. In this chapter we develop the basic aspects of the representation theory of this class of algebras. This includes both "ordinary" character theory, i.e., the theory of representations of a split semisimple algebra, and "modular" representation theory, i.e., the theory of decomposition maps.

After some general remarks about trace functions, the main results on Schur relations and orthogonality relations will be proved in Section 7.2. The subsequent section contains a discussion of integrality properties. These can be seen as generalizations of known facts about characters of finite groups, e.g., the fact that the degrees of the irreducible characters of a finite group divide the group order. We also introduce the notion of a *character table* for H; see (7.3.11).

Then Section 7.4 deals with decomposition maps. Usually, this is done for algebras over complete discrete valuation rings. But in view of our applications to Iwahori–Hecke algebras, we have to make sure that such a theory also works for more general ground rings. The main applications are Tits's Theorem 7.4.6 and the semisimplicity criterion in Theorem 7.4.7.

The modular representation theory will be developed further in Section 7.5, where we study properties of the decomposition map in more detail. These results will play a role in Chapter 11, where we describe algorithms for computing character values of Iwahori–Hecke algebras.

We make a number of general assumptions. All of our rings have identity elements, and ring homomorphisms preserve these identity elements. In particular, an algebra over a commutative ring is unital. All of our modules are right modules but endomorphisms are written on the left, unless explicitly stated otherwise.

We shall need *Wedderburn's theorem* (see [Curtis and Reiner 1981, §3B]), in the following form. Let H be a finite-dimensional algebra over a field K. The radical $\mathrm{rad}(H)$ is the nilpotent ideal consisting of all elements of H which act as 0 on each simple H-module. The algebra H is semisimple if and only if $\mathrm{rad}(H) = \{0\}$. If this is the case, there is a direct sum decomposition $H = \bigoplus_V H(V)$ where V runs over the simple H-modules (up to isomorphism), and each $H(V)$ is a simple K-algebra. This decomposition is such that an element $h \in H(V)$ acts on a simple module V' as 0 unless V and V' are isomorphic. Since $H(V)$ is a simple algebra, we have an isomorphism $\rho_V \colon H(V) \to M_{n_V}(D_V)$ onto a full matrix algebra, where D_V is a division algebra over K and n_V is the multiplicity of V as a composition factor of H regarded as a module over itself. Moreover, we have $D_V \cong \mathrm{End}_H(V)$ and $\dim_K V = n_V \dim_K D_V$.

We shall say that a simple module V is *split simple* if $\dim_K D_V = 1$, and that H is *split* if all simple modules are split simple.

In Section 7.5 we will also require some basic familiarity with non-semisimple algebras, for which we refer to Chapters 5 and 6 in [Curtis and Reiner 1981].

7.1 TRACE FUNCTIONS

We start with an associative A-algebra H where A is any commutative ring. We assume that H is finitely generated and free over A. The purpose of this section is to present the basic results about trace functions and the Gaschütz–Ikeda lemma for symmetric algebras.

7.1.1. Definition. A *trace function* on H is an A-linear map $\tau \colon H \to A$ such that $\tau(hh') = \tau(h'h)$ for all $h, h' \in H$. The set of trace functions on H is an A-module, with pointwise defined operations. We say that a trace function τ is a *symmetrizing trace* or that H is a *symmetric algebra* if the bilinear form

$$H \times H \to A, \quad (h, h') \mapsto \tau(hh'),$$

is non-degenerate, i.e., if the determinant of the matrix $(\tau(bb'))_{b,b' \in \mathcal{B}}$ is a unit in A for some (and hence every) A-basis $\mathcal{B}$ of H.

If τ is a symmetrizing trace on H and $\mathcal{B}$ is a basis for H, we denote by $\mathcal{B}^\vee = \{b^\vee \mid b \in \mathcal{B}\}$ the dual basis; it is uniquely determined by the requirement that $\tau(b^\vee b') = \delta_{bb'}$ for all $b, b' \in \mathcal{B}$.

7.1.2. Example. Let $n \geq 1$ and $H = M_n(A)$, the algebra of $n \times n$ matrices with entries in A. By Exercise 7.1, *every* trace function on H is a scalar multiple of the usual matrix trace, which we denote by Tr.

For $1 \leq a, b \leq n$ let E_{ab} be the $n \times n$ matrix with (a, b)th entry 1 and 0 otherwise. Then $\mathcal{B} = \{E_{ab} \mid 1 \leq a, b \leq n\}$ is an A-basis for H. We have

$$E_{ab}E_{a'b'} = \delta_{ba'}E_{ab'} \qquad \text{for all } 1 \leq a, b, a', b' \leq n.$$

It follows that Tr is a symmetrizing trace and the basis dual to $\mathcal{B}$ is given by $E_{ab}^\vee = E_{ba}$ for all a, b. Note that the sets $\mathcal{B}$ and $\mathcal{B}^\vee$ are equal.

7.1.3. Example. Let $H = \bigoplus_{i \in I} M_{n_i}(A)$ where I is a finite index set and n_i are positive integers for all i. Composing the projection on the ith factor with the usual matrix trace on that factor, we obtain a trace function τ_i on H. Let $d_i \in A$ ($i \in I$) and set $\tau := \sum_{i \in I} d_i \tau_i$. Then, clearly, τ is a trace function on H.

For each $i \in I$ let $\mathcal{B}_i$ be a basis for $M_{n_i}(A) \subseteq H$ as in Example 7.1.2, with dual basis $\mathcal{B}_i^\vee$ taken with respect to the usual matrix trace, i.e., the restriction of τ_i to $M_{n_i}(A)$. Then $\mathcal{B} = \coprod_i \mathcal{B}_i$ is a basis for H. We have

$$\tau(b' b^\vee) = d_i \delta_{ii'} \delta_{bb'} \qquad \text{for } b \in \mathcal{B}_i \text{ and } b' \in \mathcal{B}_{i'}.$$

It follows that τ is non-degenerate if and only if all d_i are units in A. If this is the case, the set $\bigcup_i \{d_i^{-1} b^\vee \mid b \in \mathcal{B}_i\}$ is a basis dual to $\mathcal{B}$ with respect to τ.

By using Wedderburn's theorem, this example shows that all split semisimple algebras over a field are symmetric.

7.1.4. Example. Let V be an H-module which is finitely generated and free over A. Such a module will be called an H-*lattice*. The action of H on V will be written in the form $v \mapsto vh$ ($v \in V$, $h \in H$). We obtain a corresponding algebra homomorphism

$$\rho_V \colon H \to \mathrm{End}_A(V), \quad \text{where } v\rho_V(h) = vh \text{ for } v \in V, h \in H.$$

We call ρ_V the *representation afforded by* V. The corresponding *character* is the A-linear map defined by $\chi_V \colon H \to A$, $h \mapsto \mathrm{Tr}(\rho_V(h))$, where Tr denotes the usual matrix trace. Then χ_V is indeed a trace function.

7.1.5. Example. Let G be a finite group and $H = A[G]$ the group algebra of G over A. If $h \in H$ we write $h = \sum_{g \in G} a(h)_g g$ with $a(h)_g \in A$. Then the map $\tau \colon H \to A$, $h \mapsto a(h)_1$, is a trace function. Let $\mathcal{B} := \{g \mid g \in G\}$ be the standard basis of H. Then, for all $g, h \in G$, we have $\tau(gh) = 1$ if $h = g^{-1}$ and $\tau(gh) = 0$ otherwise. It follows that τ is a symmetrizing trace, and that the dual basis is given by $\mathcal{B}^\vee = \{g^{-1} \mid g \in G\}$. Again, $\mathcal{B} = \mathcal{B}^\vee$ as sets.

7.1.6. The centre and the space of trace functions. Let τ be a symmetrizing trace on H. Then we have a duality between the centre of H and the space of trace functions on H, in the following sense.

For any $\lambda \in H^* := \mathrm{Hom}_A(H, A)$ we can uniquely define an element $\lambda^* \in H$ by the condition that $\tau(\lambda^* h) = \lambda(h)$ for all $h \in H$. The definition shows that, for example, $\tau^* = 1_H$. More generally, we have

$$\lambda^* = \sum_{b \in \mathcal{B}} \lambda(b) b^\vee, \qquad \text{where } \mathcal{B} \text{ is any } A\text{-basis of } H.$$

In order to prove such a relation, we just have to multiply both sides by an arbitrary basis element and compute the trace on the resulting elements. The above relations show that the map $H^* \to H$, $\lambda \mapsto \lambda^*$ is an A-linear isomorphism.

For any $h, h' \in H$, define their *commutator* by $[h, h'] := hh' - h'h$. Let $[H, H] \subseteq H$ be the A-submodule spanned by all commutators. By definition, an A-linear map $\lambda\colon H \to A$ is a trace function if and only if $[H, H] \subseteq \ker(\lambda)$. Thus, the A-module of trace functions on H is canonically isomorphic to the dual space of $H/[H, H]$, i.e., to $\mathrm{Hom}_A(H/[H, H], A)$.

7.1.7. Lemma. *Let $\lambda \in H^*$ and λ^* be the corresponding element in H. Then λ is a trace function if and only if λ^* lies in the centre of H. Moreover, the centre of H is a free A-submodule of H if and only if $\mathrm{Hom}_A(H/[H, H], A)$ is a free A-module.*

Note that, if $H/[H, H]$ is a free A-module then so is $\mathrm{Hom}_A(H/[H, H], A)$, but the converse is not true in general.

Proof. Let $\lambda \in H^*$. Then λ is a trace function if and only if $\lambda(hh') = \lambda(h'h)$ for all $h, h' \in H$. Using the definition of λ^*, this is equivalent to the condition that $\tau(\lambda^*(hh' - h'h)) = 0$ for all $h, h' \in H$. On the other hand, we have $\lambda^* \in Z(H)$, i.e., $\lambda^* h = h\lambda^*$ for all $h \in H$, if and only if $\tau((\lambda^* h - h\lambda^*)h') = 0$ for all $h' \in H$. Comparing these two conditions, we see that they are equivalent if and only if

$$\tau(\lambda^* h'h) = \tau(h\lambda^* h') \qquad \text{for all } h, h' \in H.$$

But this certainly holds since τ is a trace function. The map $\lambda \mapsto \lambda^*$ therefore defines an A-module isomorphism between $\mathrm{Hom}_A(H/[H, H], A)$ and the centre of H. Hence the centre is free if and only if $\mathrm{Hom}_A(H/[H, H], A)$ is free. □

7.1.8. Example. Let $H = A[G]$ be the group algebra over A of a finite group G, with symmetrizing trace as before. Let $\mathrm{Cl}(G)$ be the set of conjugacy classes of G. For each $C \in \mathrm{Cl}(G)$, we let τ_C be the indicator function on the conjugacy class C^{-1} of G. Then $\{\tau_C \mid C \in \mathrm{Cl}(G)\}$ certainly is an A-basis for the space of trace functions on H. For each C let $\hat{C}$ be the unique element in the centre of $A[G]$ such that $\hat{C} = \tau_C^*$. Writing $\hat{C} = \sum_{g \in G} a(C)_g g$ with $a(C)_g \in A$, we have

$$a(C)_g = \tau(\hat{C} \cdot g^{-1}) = \hat{C}^*(g^{-1}) = \tau_C(g^{-1}) = \begin{cases} 1 & \text{if } g \in C, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\hat{C}$ is the sum of the elements in C. Using Lemma 7.1.7, we have thus recovered the well-known result that these class sums form an A-basis for the centre of H. (We will generalize this later to construct a basis for the centre of an Iwahori–Hecke algebra; see Corollary 8.2.4.)

For the remainder of this section, we assume that H is symmetric, with symmetrizing trace τ. The existence of τ allows certain representation-theoretic constructions which we will now explain.

* **7.1.9. Definition.** Let V, V' be (right) H-modules and $\mathcal{B}$ be a basis of H, with dual basis $\mathcal{B}^\vee$. For any $\varphi \in \mathrm{Hom}_A(V, V')$ we define $I(\varphi) \in \mathrm{Hom}_A(V, V')$ by

$$I(\varphi)(v) := \sum_{b \in \mathcal{B}} \varphi(vb)b^\vee \qquad (v \in V).$$

The definition immediately implies the following rules: if V, V', V'' are H-modules and $\varphi \in \mathsf{Hom}_H(V, V')$, $\psi \in \mathsf{Hom}_A(V', V'')$, $\pi \in \mathsf{Hom}_A(V'', V)$, then

$$I(\psi \circ \varphi) = I(\psi) \circ \varphi \quad \text{and} \quad I(\varphi \circ \pi) = \varphi \circ I(\pi).$$

The following result will show that the operator $I(\varphi)$ provides a tool that turns any A-homomorphism into an H-homomorphism.

7.1.10. Lemma. *Let* $\varphi \in \mathsf{Hom}_A(V, V')$. *Then the homomorphism* $I(\varphi)$ *lies in* $\mathsf{Hom}_H(V, V')$ *and does not depend on the choice of the basis* $\mathcal{B}$.

Proof. First we check the independence of the basis. Let $\mathcal{C}$ be another basis of H, with dual basis $\{c^\vee \mid c \in \mathcal{C}\}$. We have equations $c = \sum_{b \in \mathcal{B}} \lambda_{bc} b$ where (λ_{bc}) is an invertible matrix of elements of A. For $b \in \mathcal{B}$ we have $\tau(b^\vee c) = \sum_{b'} \lambda_{b'c} \tau(b^\vee b') = \lambda_{bc}$. Hence $b^\vee = \sum_{c \in \mathcal{C}} \lambda_{bc} c^\vee$. We can now compute that

$$\sum_c \varphi(vc)c^\vee = \sum_{c,b} \lambda_{bc} \varphi(vb) c^\vee = \sum_b \varphi(vb) \Big(\sum_c \lambda_{bc} c^\vee \Big) = \sum_b \varphi(vb) b^\vee$$

for all $v \in V$, as desired. Next we check that $I(\varphi) \in \mathsf{Hom}_H(V, V')$. For any $h \in H$ and $b \in \mathcal{B}$ we write $hb = \sum_{b'} \rho(h)_{b'b} b'$ with $\rho(h)_{b'b} \in A$. Then

$$\tau(b'^\vee hb) = \sum_{b''} \rho(h)_{b''b} \tau(b'^\vee b'') = \rho(h)_{b'b},$$

and so $b'^\vee h = \sum_b \rho(h)_{b'b} b^\vee$. For any $v \in V$, we can now compute that

$$\begin{aligned} I(\varphi)(vh) &= \sum_b \varphi(vhb) b^\vee = \sum_{b,b'} \rho(h)_{b'b} \varphi(vb') b^\vee \\ &= \sum_{b'} \varphi(vb') \Big(\sum_b \rho(h)_{b'b} b^\vee \Big) = \sum_{b'} \varphi(vb') b'^\vee h = (I(\varphi)(v))h, \end{aligned}$$

and so $I(\varphi)$ commutes with the action of H. □

One of the important properties of the construction in Definition 7.1.9 lies in the fact that it provides a criterion for an H-module to be projective. Recall that an H-module V is *projective* if every epimorphism $M \xrightarrow{\pi} V$ of H-modules splits, that is, if there exists some $\iota \in \mathsf{Hom}_H(V, M)$ such that $\pi \circ \iota = \mathsf{id}_V$. For example, free H-modules are projective.

7.1.11. Lemma (Gaschütz–Ikeda). *Let V be an H-module which is projective as an A-module. Then V is projective (as an H-module) if and only if there exists some* $\varphi \in \mathsf{End}_A(V)$ *such that* $I(\varphi) = \mathsf{id}_V$.

Proof. Suppose first that there exists some $\varphi \in \mathsf{End}_A(V)$ such that $I(\varphi) = \mathsf{id}_V$. Let $M \xrightarrow{\pi} V$ be an epimorphism of H-modules. Since V is projective over A, we can find an A-linear map $\mu\colon V \to M$ such that $\pi \circ \mu = \mathsf{id}_V$. We compose this equation with φ and apply $I(\cdot)$ to both sides. The right-hand side just gives id_V. Since π is an H-module homomorphism, the left-hand side yields

$$I(\pi \circ \mu \circ \varphi) = \pi \circ I(\mu \circ \varphi).$$

Hence $\iota := I(\mu \circ \varphi) \in \mathrm{Hom}_H(V, M)$ is the desired map.

Conversely, assume that every epimorphism $M \xrightarrow{\pi} V$ splits. Let $M := V \otimes_A H$ where H is regarded as a right module over itself, and M is a right module in a natural way. Choose a basis $\mathcal{B}$ of H. Then every element $m \in M$ can be written uniquely in the form $m = \sum_{b \in \mathcal{B}} v_b \otimes b$ where $v_b \in V$. We write $1_H = \sum_{b \in \mathcal{B}} \xi_b b^\vee$ where $\xi_b \in A$, and define $\varphi \in \mathrm{End}_A(M)$ by

$$\varphi\Big(\sum_{b \in \mathcal{B}} v_b \otimes b\Big) := \sum_{b \in \mathcal{B}} \xi_b v_b \otimes 1.$$

Using relations $hb = \sum_{b'} \rho(h)_{b'b} b'$ as in the proof of Lemma 7.1.10, a straightforward computation shows that $I(\varphi) = \mathrm{id}_M$.

Now define $\pi\colon M \to V$ by $\pi(v \otimes h) := vh$. Then $\pi \in \mathrm{Hom}_H(M, V)$ and π is surjective. Since V is projective there exists some $\iota \in \mathrm{Hom}_H(V, M)$ such that $\pi \circ \iota = \mathrm{id}_V$. We set $\varphi' := \pi \circ \varphi \circ \iota \in \mathrm{End}_A(V)$. Since π and ι commute with the action of H we can now conclude

$$I(\varphi') = I(\pi \circ \varphi) \circ \iota = \pi \circ I(\varphi) \circ \iota = \pi \circ \mathrm{id}_M \circ \iota = \pi \circ \iota = \mathrm{id}_V.$$

Hence $\varphi' \in \mathrm{End}_A(V)$ has the desired property. □

7.1.12. Example. Let $H = M_n(A)$ for some $n \geq 1$ and τ be the usual trace, as in Example 7.1.2. Let $\mathcal{B} = \{E_{ab} \mid 1 \leq a, b \leq n\}$ be the standard basis of H and recall that $E_{ab}^\vee = E_{ba}$. Consider the H-lattice $V = A^{1 \times n}$ (row vectors), with H acting by matrix multiplication. Let $(e_1, \ldots, e_n)$ be the standard basis of V. Let $\varphi \in \mathrm{End}_A(V)$ be such that $\varphi(e_b) = \sum_c \varphi_{cb} e_c$ where $\varphi_{cb} \in A$. Now we can compute that

$$\begin{aligned} I(\varphi)(e_a) &= \sum_{a',b'} \varphi(e_a E_{a'b'}) E_{b'a'} = \sum_{a',b'} \varphi(\delta_{aa'} e_{b'}) E_{b'a'} \\ &= \sum_{b'} \Big(\sum_c \varphi_{cb'} e_c\Big) E_{b'a} = \sum_{b',c} \varphi_{cb'} e_c E_{b'a} = \sum_{b',c} \varphi_{cb'} \delta_{b'c} e_a \\ &= \mathrm{Tr}(\varphi) e_a. \end{aligned}$$

Hence, choosing any $\varphi \in \mathrm{End}_A(V)$ with $\mathrm{Tr}(\varphi) = 1$, we see that $I(\varphi) = \mathrm{id}_V$. In particular, V is projective.

7.1.13. Example. Let G be a finite group and $H = A[G]$ the group algebra over A, with symmetrizing trace τ as in Example 7.1.5. Recall that the basis dual to $\mathcal{B} = \{g \mid g \in G\}$ is $\mathcal{B}^\vee = \{g^{-1} \mid g \in G\}$. For any H-lattice V, we have

$$I(\mathrm{id}_V)(v) = \sum_{g\in G} \mathrm{id}_V(vg)g^{-1} = \sum_{g\in G} v = |G|\, v \qquad (v \in V).$$

Hence, if $|G|1_A \in A$ is a unit, then every H-lattice is projective. In particular, if A is a field whose characteristic does not divide $|G|$ then $A[G]$ is a semisimple algebra. Thus, we have recovered *Maschke's theorem* (see [Isaacs 1976, (1.9)]).

7.2 SCHUR RELATIONS AND SCHUR ELEMENTS

Let H be a finite-dimensional algebra over a field K and assume that we are given a symmetrizing trace τ on H. We will now see that the existence of τ implies that we have orthogonality relations for the irreducible characters of H. The main tool will be the operator $I(\cdot)$ of Definition 7.1.9.

Any H-module will be tacitly assumed to be finite-dimensional. Lemma 7.1.10 shows that if V, V' are H-modules such that $\mathrm{Hom}_H(V, V') = 0$ then $I(\varphi) = 0$ for all $\varphi \in \mathrm{Hom}_K(V, V')$. On the other hand, if $V = V'$ is a simple module, we know by *Schur's lemma* that $\mathrm{End}_H(V)$ is a division algebra over K. Recall that V is split simple if $\mathrm{End}_H(V) = K \cdot \mathrm{id}_V$.

7.2.1. Theorem. *Let V be a split simple H-module. Then there is a unique element $c_V \in K$ such that*

$$I(\varphi) = c_V \,\mathrm{Tr}(\varphi)\, \mathrm{id}_V \qquad \text{for all } \varphi \in \mathrm{End}_K(V).$$

Furthermore, the constant c_V only depends on the isomorphism class of V.

The element $c_V \in K$ will be called the *Schur element* associated with V.

Proof. The uniqueness of c_V is clear since there certainly exists some endomorphism of V with trace 1. Now let $\varphi \in \mathrm{End}_K(V)$. By Lemma 7.1.10 we have $I(\varphi) \in \mathrm{End}_H(V)$. Since V is split simple, we have $I(\varphi) = c_{V,\varphi}\,\mathrm{id}_V$ for some $c_{V,\varphi} \in K$. We first consider special choices for φ for which we can explicitly compute these constants.

Assume that $\dim_K V = n$ and choose a basis $(v_1, \ldots, v_n)$. For $1 \leq i, j \leq n$ let $\varphi_{ij} \in \mathrm{End}_K(V)$ such that $\varphi_{ij}(v_l) = \delta_{il} v_j$. To simplify notation, we write c_{ij} instead of $c_{V,\varphi_{ij}}$. For $h \in H$ let also $v_l h = \sum_k \rho(h)_{kl} v_k$ where $\rho(h)_{kl} \in K$. Using this notation, a straightforward computation yields that

$$I(\varphi_{ij})(v_l) = \sum_{b,k} \rho(b)_{il}\rho(b^\vee)_{kj} v_k.$$

Since $I(\varphi_{ij}) = c_{ij}\,\mathrm{id}_V$ we conclude that

$$\sum_b \rho(b)_{il}\rho(b^\vee)_{kj} = \delta_{kl}c_{ij} \qquad \text{for all } i,j,k,l.$$

The dual basis of $\mathcal{B}^\vee$ is again $\mathcal{B}$. By Lemma 7.1.10, we can therefore interchange the roles of $\mathcal{B}$ and its dual basis in the above computation without affecting the result. Thus, we also obtain

$$\sum_b \rho(b^\vee)_{i'l'}\rho(b)_{k'j'} = \delta_{k'l'}c_{i'j'} \qquad \text{for all } i',j',k',l'.$$

Now we set $i' = k$, $l' = j$, $k' = i$ and $j' = l$. Then the left-hand sides of the last two equations are the same. Hence so are the right-hand sides and we have reached the conclusion that

$$\delta_{kl}c_{ij} = \delta_{ij}c_{kl} \qquad \text{for all } i,j,k,l.$$

We set $k = l = 1$ and $c_V := c_{11}$. If $i \neq j$ then $c_{ij} = \delta_{ij}c_V = 0$, and if $i = j$ then $c_{ii} = c_{11} = c_V$. This implies that $I(\varphi_{ij}) = \delta_{ij}c_V \,\mathrm{id}_V$. Since $\mathrm{Tr}(\varphi_{ij}) = \delta_{ij}$, this proves the assertion for all endomorphisms φ_{ij}. The defining formula in Definition 7.1.9 shows that the assignment $\varphi \mapsto c_{V,\varphi}$ is linear in φ. Since the φ_{ij} form a basis of $\mathrm{End}_K(V)$, we conclude that for any $\varphi \in \mathrm{End}_K(V)$ we have $I(\varphi) = c_V \,\mathrm{Tr}(\varphi)\,\mathrm{id}_V$.

It remains to show that $c_V = c_{V'}$ if V' is an H-module isomorphic to V. Let $\sigma\colon V' \to V$ be an H-module isomorphism and $\varphi' := \sigma^{-1} \circ \varphi \circ \sigma \in \mathrm{End}_K(V')$. Since σ commutes with the action of H we can compute that

$$c_{V'}\,\mathrm{Tr}(\varphi')\,\mathrm{id}_{V'} = I(\varphi') = \sigma^{-1} \circ I(\varphi) \circ \sigma = c_V\,\mathrm{Tr}(\varphi)\,\mathrm{id}_V.$$

Since certainly $\mathrm{Tr}(\varphi') = \mathrm{Tr}(\varphi)$, this implies $c_{V'} = c_V$. □

For some purposes, it may be convenient to reformulate these relations in terms of matrix representations and characters.

7.2.2. Corollary (Schur relations). *Let V, V' be split simple H-modules affording matrix representations $\rho\colon H \to M_n(K)$ and $\rho'\colon H \to M_m(K)$, respectively, where $n = \dim_K V$ and $m = \dim_K V'$. Then we have*

$$\sum_{b\in\mathcal{B}} \rho(b)_{il}\rho'(b^\vee)_{kj} = \begin{cases} \delta_{ij}\delta_{kl}c_V & \text{if } V = V' \text{ and } \rho = \rho', \\ 0 & \text{if } V \not\cong V', \end{cases}$$

for all $1 \leq i,l \leq n$, $1 \leq j,k \leq m$.

Proof. Choose bases $(v_1, \ldots, v_n)$ and $(v'_1, \ldots, v'_m)$ of V and V' defining the representations ρ and ρ', respectively. For $1 \leq i \leq n$ and $1 \leq j \leq m$ let $\varphi_{ij} \in \mathrm{Hom}_K(V, V')$ be defined by $\varphi_{ij}(v_l) = \delta_{il}v'_j$. The above relations are then obtained by writing out the equations $I(\varphi_{ij}) = 0$ (if $V \not\cong V'$) and $I(\varphi_{ij}) = \delta_{ij}c_V\,\mathrm{id}_V$ (if $V = V'$ and $\rho = \rho'$). □

We can now prove a kind of converse to Theorem 7.2.1:

7.2.3. Remark. Let V be an H-module and $0 \neq c \in K$ be such that $I(\varphi) = c\,\mathrm{Tr}(\varphi)\,\mathrm{id}_V$ for all $\varphi \in \mathrm{End}_K(V)$. Then V is split simple and $c = c_V$.

Proof. Assume, if possible, that V is not simple. Choosing a basis adapted to a non-trivial proper invariant subspace of V, we obtain a matrix representation $\rho\colon H \to M_n(K)$ afforded by V such that $\rho(h)_{n1} = 0$ for all $h \in H$. Writing out the equation $I(\varphi) = c\,\mathrm{Tr}(\varphi)\,\mathrm{id}_V$ as in the proof of Corollary 7.2.2 and choosing $k = l = n$ and $i = j = 1$ yields that $0 = c$, contrary to our assumption. Hence V is simple. Now consider any $\varphi \in \mathrm{End}_H(V)$. Take $\varphi' \in \mathrm{End}_K(V)$ with $\mathrm{Tr}(\varphi') = 1$. Then we compute that

$$c\,\mathrm{Tr}(\varphi \circ \varphi')\,\mathrm{id}_V = I(\varphi \circ \varphi') = \varphi \circ I(\varphi') = c\,\mathrm{Tr}(\varphi')\varphi = c\varphi.$$

We conclude that φ is a scalar multiple of id_V. Hence V is split simple. □

7.2.4. Corollary (Orthogonality relations). *Let V and V' be split simple H-modules and denote by χ_V and $\chi_{V'}$ their characters, respectively. Then*

$$\sum_{b \in \mathcal{B}} \chi_V(b)\chi_{V'}(b^\vee) = \begin{cases} c_V \dim_K V & \text{if } \chi_V = \chi_{V'}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. First note that we have $\chi_V = \chi_{V'}$ if and only if $V \cong V'$. (This follows easily from Exercise 7.4.) Using the notation of the proof of Corollary 7.2.2, we have $\chi_V(h) = \sum_i \rho(h)_{ii}$ and $\chi_{V'}(h) = \sum_j \rho(h)'_{jj}$ for all $h \in H$. It remains to insert this into the above expression and to evaluate using the Schur relations. □

7.2.5. Example. Let $H = K[G]$ be a group algebra with symmetrizing trace τ as in Example 7.1.5. Assume that K is a field of characteristic 0. Let V be a split simple H-module. Then $I(\mathrm{id}_V) = c_V(\dim_K V)\,\mathrm{id}_V$. On the other hand, we have seen in Example 7.1.13 that $I(\mathrm{id}_V) = |G|\,\mathrm{id}_V$. Hence we have $c_V = |G|/\dim_K V$. Combining this with Corollary 7.2.4, we obtain the well-known orthogonality relations for the (absolutely) irreducible characters of G.

Combining the Gaschütz–Ikeda Lemma 7.1.11 with the above results on the Schur relations, we obtain the following semisimplicity criterion.

7.2.6. Theorem. *A split simple H-module V is projective if and only if $c_V \neq 0$. In particular, assuming that H is split, H is a semisimple algebra if and only if all Schur elements are non-zero. If this is the case, we have*

$$\tau = \sum_V \frac{1}{c_V}\chi_V$$

where V runs over the simple H-modules (up to isomorphism).

Proof. Let V be a split simple H-module. Assume first that $c_V \neq 0$. Then we can take any $\varphi \in \mathsf{End}_K(V)$ with $\mathsf{Tr}(\varphi) = c_V^{-1}$ (which certainly exists) and obtain $I(\varphi) = c_V \mathsf{Tr}(\varphi)\,\mathsf{id}_V = \mathsf{id}_V$. Hence Lemma 7.1.11 shows that V is projective. Conversely, assume that V is projective. Again by using Lemma 7.1.11, there exists some $\varphi \in \mathsf{End}_K(V)$ such that $I(\varphi) = \mathsf{id}_V$. Hence $\mathsf{Tr}(\varphi)c_V = 1$, and so $c_V \neq 0$. Thus, assuming that H is split, all Schur elements are non-zero if and only if all simple H-modules are projective, which is certainly equivalent to H being semisimple.

It remains to prove the assertion about the expression for τ. Assume that H is split and semisimple. Let $H = \bigoplus_V H(V)$ be the Wedderburn decomposition of H (see the introduction to this chapter), where V runs over the simple H-modules (up to isomorphism). For each V there is an isomorphism $\rho_V \colon H(V) \to M_{n_V}(K)$, where $n_V = \dim_K V$.

By Example 7.1.2, the restriction of τ to $H(V)$ is of the form $d_V \mathsf{Tr}_V \circ \rho_V$ for some $d_V \in K$, where Tr_V denotes the usual trace on $M_{n_V}(K)$. Now note that the composition of the projection $H \to H(V)$ with $\mathsf{Tr}_V \circ \rho_V$ just equals the character χ_V. Thus we have $\tau = \sum_V d_V \chi_V$, and we must show that $d_V c_V = 1$ for all V. Since τ is non-degenerate, we have $d_V \neq 0$ for all V; see Example 7.1.3.

Let $\mathcal{B} = \coprod_V \mathcal{B}(V)$ be a basis of H which is adapted to the decomposition $H = \bigoplus_V H(V)$. For each V, let $\tilde{\mathcal{B}}(V)$ be the basis dual to $\mathcal{B}(V)$ taken with respect to the trace function $\mathsf{Tr}_V \circ \rho_V$ on $H(V)$. Then $\coprod_V \{d_V^{-1}\tilde{b} \mid \tilde{b} \in \tilde{\mathcal{B}}(V)\}$ is the basis of H which is dual to $\mathcal{B}$ with respect to the form τ; see again Example 7.1.3.

Now fix a simple module V, an endomorphism $\varphi \in \mathsf{End}_K(V)$ and a vector $v \in V$. We consider the defining equation for c_V:

$$c_V \mathsf{Tr}(\varphi)v = I(\varphi)(v) = \sum_{b\in\mathcal{B}} \varphi(vb)b^\vee = \sum_{V'} \sum_{b\in\mathcal{B}(V')} \varphi(vb)b^\vee.$$

Since $h \in H(V')$ acts as 0 on V unless V and V' are isomorphic, we can restrict the above sum to $\mathcal{B}(V)$ and obtain

$$c_V \mathsf{Tr}(\varphi)v = \sum_{b\in\mathcal{B}(V)} \varphi(vb)b^\vee = d_V^{-1} \sum_{b\in\mathcal{B}(V)} \varphi(vb)\tilde{b}.$$

The summation on the right-hand side is the defining formula for the Schur element of V regarded as a module for $H(V)$ with symmetrizing trace $\mathsf{Tr}_V \circ \rho_V$. Example 7.1.12 shows that this Schur element equals 1, and so the right-hand side evaluates to $d_V^{-1}\mathsf{Tr}(\varphi)v$. This holds for all $v \in V$ and all $\varphi \in \mathsf{End}_K(V)$. Hence we conclude that $c_V = d_V^{-1}$. □

The Schur relations can also be used to obtain explicit formulas for idempotents in H. For this purpose, assume that H is split semisimple and let $H = \bigoplus_V H(V)$ be the Wedderburn decomposition of H, where V runs over the simple

H-modules (up to isomorphism). For each simple H-module V, we have an isomorphism $\rho_V\colon H(V) \to M_{n_V}(K)$ where $n_V = \dim_K V$. Let $\mathcal{B}$ be any basis of H. By Theorem 7.2.6, the Schur element c_V is non-zero, and we set

$$e_{ij}^V := \frac{1}{c_V} \sum_{b \in \mathcal{B}} \rho_V(b)_{ji} b^\vee, \qquad 1 \leq i, j \leq n_V.$$

7.2.7. Proposition (Formulas for idempotents). *Recall that* H *is assumed split and semisimple. Then, in the above set-up, the following hold.*

(a) *We have $\rho_{V'}(e_{ij}^V) = 0$ if $V \not\cong V'$ and $\rho_V(e_{ij}^V)$ is the matrix with (i,j)th coefficient 1 and coefficient 0 otherwise. Hence, $\{e_{ij}^V\}$ is a basis of* H.

(b) *We have $1_H = \sum_V \sum_{i=1}^{n_V} e_V^{ii}$ and this is a decomposition of 1_H into orthogonal primitive idempotents. Moreover, for fixed V, the idempotents e_{ii}^V $(1 \leq i \leq n_V)$ are all conjugate by units in* H.

(c) *Let $1_H = \sum_V e_V$ with $e_V \in H(V)$. Then each e_V is a central primitive idempotent and, denoting by χ_V the character of* V, *we have*

$$e_V = \sum_{i=1}^{n_V} e_{ii}^V = \frac{1}{c_V} \sum_{b \in \mathcal{B}} \chi_V(b) b^\vee.$$

Proof. The statement in (a) immediately follows using the Schur relations in Corollary 7.2.2. We can now assume that $H = \bigoplus_V M_{n_V}(K)$ and that, in the direct summand corresponding to V, the element e_{ij}^V is the matrix with (i,j)th coefficient 1 and coefficient 0 otherwise. From this description, it is obvious that the e_{ii}^V are mutually orthogonal idempotents and that $1_H = \sum_V \sum_{i=1}^{n_V} e_{ii}^V$. Moreover, for fixed V, two matrices e_{ii}^V and e_{jj}^V are certainly conjugate by an invertible matrix in $M_{n_V}(K)$. Finally, each e_{ii}^V is primitive, since $\dim_K(e_{ii}^V M_{n_V}(K) e_{ii}^V) = 1$. Thus, all the statements in (b) are proved. Now consider (c). The element $\sum_{i=1}^{n_V} e_{ii}^V$ is the identity matrix in $M_{n_V}(K)$ and so must be equal to e_V (since the sum of all these elements is 1_H). The primitivity just expresses the fact that $H(V)$ cannot be decomposed further as a direct sum of ideals in H. The last equality in the formula for e_V holds since $\chi_V(h) = \sum_{i=1}^{n_V} \rho_V(h)_{ii}$ for any $h \in H$. □

Finally, we obtain yet another characterization of the Schur elements. Consider a split simple H-module V, with character χ_V. If $z \in H$ lies in the centre of H then z commutes with the action of any element of H on V. Hence Schur's lemma implies that z acts as a scalar on V; we denote this scalar by $\omega_V(z)$. Thus, we have defined a K-algebra homomorphism $\omega_V\colon Z(H) \to K$, which we call the *central character* associated with V.

7.2.8. Proposition. *Assume that* H *is split semisimple. For any simple* H-*module V, let $\chi_V^* \in Z(H)$ be the unique element such that $\tau(\chi_V^* h) = \chi_V(h)$ for all $h \in H$ (as in Lemma 7.1.7). Then the following hold:*

(a) *We have $\omega_{V'}(\chi_V^*) = 0$ if V, V' are not isomorphic, and $\omega_V(\chi_V^*) = c_V$.*

(b) *The elements* $\{\chi_V^*\}$*, where* V *runs over the simple* H*-modules (up to isomorphism), form a* K*-basis of* $Z(H)$.

(c) *We have* $\chi_V^* = c_V e_V$*, where* $e_V \in H$ *is the central idempotent associated with* V *as in Proposition 7.2.7.*

Proof. Let $\mathcal{B}$ be any K-basis of H. Then the definition of χ_V^* shows that

$$\chi_V^* = \sum_{b \in \mathcal{B}} \chi_V(b) b^\vee.$$

Comparison with the formula in Proposition 7.2.7 already proves (c). Since e_V acts as 0 on V' if V, V' are not isomorphic, this proves $\omega_{V'}(e_V) = 0$, and hence also $\omega_{V'}(\chi_V^*) = 0$, unless V, V' are isomorphic. Now consider the formula $\tau = \sum_V c_V^{-1} \chi_V$ of Corollary 7.2.6. Taking "duals", it is equivalent to the formula $1_H = \sum_V c_V^{-1} \chi_V^*$. Applying $\omega_{V'}$ yields that $1 = \omega_{V'}(1_H) = \omega_{V'}(\chi_{V'}^*) c_{V'}^{-1}$. This completes the proof of (a).

Now consider (b). By Lemma 7.1.7, the correspondence $\lambda \mapsto \lambda^*$ is a K-linear isomorphism between the space of trace functions on H and the centre of H. Hence, (b) holds if and only if the set $\{\chi_V\}$ is a basis of the space of trace functions on H, which in turn holds since H is split and semisimple (see Exercise 7.4). □

7.3 GROTHENDIECK GROUPS AND INTEGRALITY

We now combine the settings of the previous two sections. Assume that A is an integral domain and K is a field containing A. Let H be an A-algebra, finitely generated and free over A.

We denote by $KH = H \otimes_A K$ the K-algebra obtained by extension of scalars from A to K. We may naturally consider H as a subset of KH. A similar convention will be applied to any ring B such that $A \subseteq B \subseteq K$, where we have natural inclusions $H \subseteq BH \subseteq KH$.

In what follows, it will be convenient to use the language of *Grothendieck groups* (see, for example, [Curtis and Reiner 1981, Section 16B]). Let $R_0(KH)$ be the Grothendieck group of the category of finite-dimensional KH-modules. Thus, $R_0(KH)$ is generated by expressions $[V]$, one for each KH-module V (up to isomorphism), with relations $[V] = [V'] + [V'']$ for each short exact sequence $0 \to V' \to V \to V'' \to 0$ of KH-modules. Two KH-modules V, V' give rise to the same element in $R_0(KH)$ if and only if V and V' have the same composition factors, counting multiplicities. It follows that $R_0(KH)$ is free abelian, with basis given by the isomorphism classes of simple modules. Finally, let $R_0^+(KH)$ be the subset of $R_0(KH)$ consisting of elements $[V]$ where V is a KH-module. This is a monoid whose identity element is the class of the 0-module.

7.3.1. Definition. Let X be an indeterminate over K and $\mathrm{Maps}(H, K[X])$ the K-algebra of maps from H to $K[X]$ (with pointwise multiplication of maps as algebra multiplication). We define a map