NOTES FOR 128: COMBINATORIAL REPRESENTATION THEORY OF COMPLEX LIE ALGEBRAS AND RELATED TOPICS

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Contents

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Recommended reading

[Bou] N. Bourbaki, *Elements of Mathematics: Lie Groups and Algebras.*

Not always easy to read from front to back, but it was clearly written by the oracles of mathematics at the time, with the purpose of containing everything.

[FH] W. Fulton, J. Harris, *Representation Theory: A first course.*

Written for the non-specialist, but rich with examples and pictures. Mostly, an example-driven tour of finite-dimensional representations of finite groups and Lie algebras and groups. Cheap – buy this book.

[Hum] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory.*

Lightweight approach to finite-dimensional Lie algebras. Has a lot of the proofs, but not a lot of examples.

[Ser] J. J. Serre, *Complex Semisimple Lie Algebras.*

Super lightweight. A tour of the facts, without much proof, but great quick reference.

1. The poster child of CRT: the symmetric group

Combinatorial representation theory is the study of representations of algebraic objects, using combinatorics to keep track of the relevant information. To see what I mean, let's take a look at the symmetric group.

Let F be your favorite field of characteristic 0. Recall that an *algebra* A over F is a vector space over F with an associative multiplication

$$A \otimes A \to A$$

Here, the tensor product is over F, and just means that the multiplication is bilinear. Our favorite examples for a while will be

(1) Group algebras (today)
(2) Enveloping algebras of Lie algebras (tomorrow-ish)

And our favorite field is $F = \mathbb{C}$.

The *symmetric group* S_k is the group of permutations of $\{1, \dots, k\}$. The *group algebra* $\mathbb{C}S_k$ is the vector space

$$\mathbb{C}S_k = \left\{ \sum_{\sigma \in S_k} c_\sigma \sigma \mid c_\sigma \in \mathbb{C} \right\}$$

with multiplication linear and associative by definition:

$$\left(\sum_{\sigma \in S_k} c_\sigma \sigma \right) \left(\sum_{\pi \in S_k} d_\pi \pi \right) = \sum_{\sigma, \pi \in G} (c_\sigma d_\pi)(\sigma\pi).$$

Example. When $k = 3$,

$$S_3 = \{1, (12), (23), (123), (132), (13)\} = \langle s_1 = (12), s_2 = (23) \mid s_1^2 = s_2^2 = 1, s_1 s_2 s_1 = s_2 s_1 s_2 \rangle.$$

So

$$\mathbb{C}S_3 = \{c_1 + c_2(12) + c_3(23) + c_4(123) + c_5(132) + c_6(13) \mid c_i \in \mathbb{C}\}$$

and, for example,

$$\begin{aligned}(2+(12))(5(123)-(23)) &= 10(123)-2(23)+5(12)(123)-(12)(23)\\ &= 10(123)-2(23)+5(23)-(123) = \boxed{3(23)+9(123)}.\end{aligned}$$

1.1. **Our best chance of understanding big bad algebraic structures: representations!** A *homomorphism* is a structure-preserving map. A *representation* of an F-algebra A is a vector space V over F, together with a homomorphism

$$\rho : A \to \mathrm{End}(V) = \{\ F\text{-linear maps } V \to V\ \}.$$

The map (equipped with the vector space) is the representation; the vector space (equipped with the map) is called an *A-module.*

Example. Favorite representation of S_n is the permutation representation: Let $V = \mathbb{C}^k = \mathbb{C}\{v_1, \dots, v_k\}$. Define

$$\rho : S_k \to \mathrm{GL}_k(\mathbb{C}) \qquad \text{by} \qquad \rho(\sigma)v_i = v_{\sigma(i)}$$

$k = 2$:

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (12) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\rho(\mathbb{C}S_2) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \ \middle|\ a, b \in \mathbb{C} \right\} \subset \mathrm{End}(\mathbb{C}^2)$$

$k = 3$:

$$1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (12) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (23) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$(123) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad (132) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad (13) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\rho(\mathbb{C}S_3) = \left\{ \begin{pmatrix} a+c & b+e & d+f \\ b+d & a+f & c+e \\ e+f & c+d & a+b \end{pmatrix} \ \middle|\ a, b, c, d, e, f \in \mathbb{C} \right\} \subset \mathrm{End}(\mathbb{C}^3)$$

A representation/module V is *simple* or *irreducible* if V has no invariant subspaces.

Example. The permutation representation is not simple since $v_1 + \cdots + v_k = (1, \dots, 1)$ is invariant, and so $T = \mathbb{C}\{(1, \dots, 1)\}$ is a submodule (called the *trivial representation*). However, the trivial representation is one-dimensional, and so is clearly simple. Also, the orthogonal compliment of T, given by

$$S = \mathbb{C}\{v_2 - v_1, v_3 - v_1, \dots, v_k - v_1\}$$

is also simple (called the *standard representation*). So V *decomposes* as

$$V = T \oplus S \tag{1.1}$$

by the change of basis

$$\{v_1, \dots, v_k\} \to \{v, w_2, \dots, w_k\} \qquad \text{where } v = v_1 + \cdots + v_k \text{ and } w_i = v_i - v_1.$$

New representation looks like

$$\rho(\sigma)v = v, \qquad \rho(\sigma)w_i = w_{\sigma(i)} - w_{\sigma(1)} \quad \text{where } w_1 = 0.$$

For example, when $k = 3$,

$$1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (12) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \qquad (23) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$(123) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad (132) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix} \qquad (13) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix}$$

Notice, the vector space $\mathrm{End}(\mathbb{C}^2)$ is four-dimensional, and the four matrices

$$\rho_S(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \rho_S((12)) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix},$$
$$\rho_S((23)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \rho_S((132)) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

are linearly independent, so $\rho_S(\mathbb{C}S_3) = \mathrm{End}(\mathbb{C}^2)$, and so (at least for $k = 3$) S is also simple! So the decomposition in (1.1) is complete.

An algebra is *semisimple* if all of its modules decompose into the sum of simple modules.

Example. The group algebra of a group G over a field F is semisimple iff $\mathrm{char}(F)$ does not divide $|G|$. So group algebras over $\mathbb{C}$ are all semisimple.

We like semisimple algebras because they are isomorphic to a direct sum over their simple modules of the ring of endomorphisms of those module (*Artin-Wedderburn theorem*).

$$A \cong \bigoplus_{V \in \hat{A}} \mathrm{End}(V)$$

where $\hat{A}$ is the set of representative of A-modules. So studying a semisimple algebra is "the same" as studying its simple modules.

Theorem 1.1. *For a finite group G, the irreducible representations of G are in bijection with its conjugacy classes.*

Proof.

(A) Show

(1) the class sums of G, given by

$$\left\{ \sum_{h \in \mathcal{K}} h \mid \mathcal{K} \text{ is a conjugacy class of } G \right\}$$

form a basis for $Z(FG)$;

Example: $G = S_3$. The class sums are

$$1, \quad (12) + (23) + (13), \quad \text{and} \quad (123) + (132)$$

(2) and $\dim(Z(FG)) = |\hat{G}|$ where $\hat{G}$ is an indexing set of the irreducible representations of G.

(B) Use character theory. A *character* χ of a group G corresponding to a representation ρ is a linear map

$$\chi_\rho : G \to \mathbb{C} \qquad \text{defined by} \qquad \chi_\rho : g \to \operatorname{tr}(\rho(g)).$$

Nice facts about characters:

(1) They're *class functions* since

$$\chi_\rho(hgh^{-1}) = \operatorname{tr}(\rho(hgh^{-1})) = \operatorname{tr}(\rho(h)\rho(g)\rho(h)^{-1}) = \operatorname{tr}(\rho(g)) = \chi_\rho(g).$$

Example. The character associated to the trivial representation of any group G is $\chi_1 = 1$.

Example. Let χ be the character associate to the standard representation of S_3. Then

$$\chi(1) = 2, \qquad \chi((12)) = \chi((23)) = \chi((13)) = 0, \qquad \chi((123)) = \chi(132) = -1.$$

(2) They satisfy nice relations like

$$\begin{aligned} \chi_{\rho\oplus\psi} &= \chi_\rho + \chi_\psi \\ \chi_{\rho\otimes\psi} &= \chi_\rho \chi_\psi \end{aligned}$$

(3) The characters associated to the irreducible representations form an orthonormal basis for the class functions on G. (This gives the bijection)

Studying the representation theory of a group is "the same" as studying the character theory of that group.

This is not a particularly satisfying bijection, either way. It doesn't say "given representation X, here's conjugacy class Y, and vice versa." $\square$

Conjugacy classes of the symmetric group are given by cycle type. For example the conjugacy classes of S_4 are

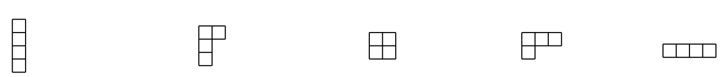
$$\begin{aligned} &\{1\} = \{(a)(b)(c)(d)\} \\ &\{(12),(13),(14),(23),(24),(34)\} = \{(ab)(c)(d)\} \\ &\{(12)(34),(13)(24),(14)(23)\} = \{(ab)(cd)\} \\ &\{(123),(124),(132),(134),(142),(143),(234),(243)\} = \{(abc)(d)\} \\ &\{(1234),(1243),(1324),(1342),(1423),(1432)\} = \{(abcd)\}. \end{aligned}$$

Cycle types of permutations of k are in bijection with *partitions* $\lambda \vdash k$:

$$\lambda = (\lambda_1, \lambda_2, \dots) \quad \text{with } \lambda_1 \geq \lambda_2 \geq \dots, \quad \lambda_i \in \mathbb{Z}_{\geq 0}, \lambda_1 + \lambda_2 + \dots = k.$$

The cycle types and their corresponding partitions of 4 are

$$\begin{array}{ccccc} (a)(b)(c)(d) & (ab)(c)(d) & (ab)(cd) & (abc)(d) & (abcd) \\ \\ (1,1,1,1) & (2,1,1) & (2,2) & (3,1) & (4) \end{array}$$

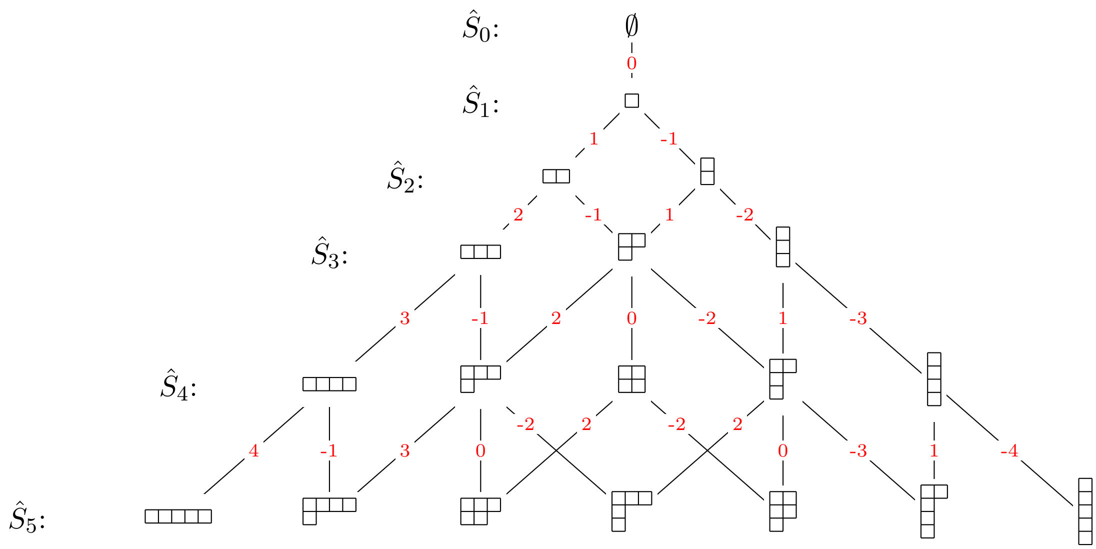


where the picture is an up-left justified arrangement of boxes with λ_i boxes in the ith row.

The combinatorics goes way deep! *Young's Lattice* is an infinite leveled labeled graph with vertices and edges as follows.

Vertices: Label vertices in label vertices on level k with partitions of k.

FIGURE 1. Young's lattice, levels 0–5.



Edges: Draw and edge from a partition of k to a partition of $k+1$ if they differ by a box.
See Figure 1.

Some combinatorial facts: (without proof)

(1) The representations of S_k are indexed by the partitions on level k.
(2) The basis for the module corresponding to a partition λ is indexed by downward-moving paths from $\emptyset$ to λ.
(3) The representation is encoded combinatorially as well. Define the *content* of a box b in row i and column j of a partition as

$$c(b) = j - i, \qquad \text{the diagonal number of } b.$$

Label each edge in the diagram by the content of the box added. The matrix entries for the transposition $(i\ i+1)$ are functions of the values on the edges between levels $i-1$, i, and $i+1$.
(4) If S^λ is the module indexed by λ, then

$$\mathrm{Ind}_{S_k}^{S_{k+1}}(S^\lambda) = \bigoplus_{\substack{\mu \vdash k+1 \\ \lambda - \mu}} S^\mu \qquad \text{and } \mathrm{Res}_{S_{k-1}}^{S_k}(S^\lambda) = \bigoplus_{\substack{\mu \vdash k-1 \\ \mu - \lambda}} S^\mu$$

(where $\mathrm{Res}_{S_{k-1}}^{S_k}(S^\lambda)$ means forget the action of elements not in S_{k-1}, and $\mathrm{Ind}_{S_k}^{S_{k+1}}(S^\lambda) = \mathbb{C}S_{k_1} \otimes_{\mathbb{C}S_k} S^\lambda$).

1.2. **Where is this all going?** Really, where has this all gone? The symmetric group is so nice in so many ways, that we've chased these combinatorial features down many paths.

One path is the categorization of other reflection groups, both finite and not. That took us to their deformations, called Hecke algebras, and other spin-off Hecke-like algebras and diagram algebras.

Another came from Schur-Weyl duality, which showed that the representation theory of S_k as k ranges, is in duality with the representation theory of $\mathrm{GL}_n(\mathbb{C})$. Then, later, people got into Lie algebras, and saw that the same results held there, and that combinatorics controls most of complex Lie theory as well. Further, there are lots of important deformations of Lie algebras whose combinatorics is also controlled combinatorially.

2. Lie algebras

A *Lie algebra* is a vector space $\mathfrak{g}$ over F with a bracket $[,] : \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$ (the tensor product implies that $[,]$ is bilinear) satisfying

(a) (*skew symmetry*) $[x, y] = -[y, x]$, and
(b) (*Jacobi identity*) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$,

for all $x, y, z \in \mathfrak{g}$. Note that a Lie algebra is not an algebra ("Lie" is not an adjective), as algebras A are vector spaces with a product under which A becomes a associative ring with identity.

For the rest of time, our favorite field is $F = \mathbb{C}$.

2.1. **Favorite Examples.**

(1) $M_n(\mathbb{C}) = \{n \times n \text{ matrices}\}$ is a Lie algebra with bracket $[x, y] = xy - yx$. This is the Lie algebra $\mathfrak{gl}_n$. This Lie algebra is associated with, but not the same thing as, *general linear group*

$$\mathrm{GL}_n(\mathbb{C}) = \{X \in M_n(\mathbb{C}) \mid X \text{ is invertible } \}.$$

Alternatively, let V be a vector space. Then

$$\mathfrak{gl}(V) = \mathrm{End}(V), \qquad \text{with bracket } [x, y] = xy - yx.$$

(2) For $V = \mathbb{C}^n$, define

$$\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{sl}(V) = \{x \in \mathfrak{gl}(V) \mid \mathrm{tr}(x) = 0\}.$$

This is associated to, but is not the same thing as, $\mathrm{SL}(V) = \{X \in \mathrm{GL}(V) \mid \det(X) = 1\}$, the *special linear group*.
Example: $n = 2$. If $V = \mathbb{C}^2$, then

$$\mathfrak{sl}(V) = \mathfrak{sl}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{C} \right\}$$

So $\dim(\mathfrak{sl}_2) = 3$, and has basis

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The relations are

$$\begin{aligned}
[h,x] = hx - xh &= \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} - \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\\
&= \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} - \begin{pmatrix}0 & -1\\ 0 & 0\end{pmatrix} = \boxed{2x}\\
[h,y] = hy - yh &= \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\\
&= \begin{pmatrix}0 & 0\\ -1 & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} = \boxed{-2y}\\
[x,y] = xy - yx &= \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\\
&= \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = \boxed{h}.
\end{aligned}$$

So another definition of $\mathfrak{sl}_2$ is the Lie algebra generated by x, y, and h, with relations

$$[x,h] = -2x, \qquad [y,h] = 2y, \qquad \text{and} \qquad [x,y] = h.$$

Since $\mathfrak{gl}_2$ is only one more dimension, choose the basis x, y, and h as above, and

$$I = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Since $I \in Z(\mathrm{End}(\mathbb{C}^2))$ we have

$$[x,I] = [y,I] = [h,I] = 0,$$

So as a Lie algebra $\mathfrak{gl}_2 \cong \mathbb{C} \oplus \mathfrak{sl}_2$. In general, $\mathfrak{gl}_n$ is one more dimension than $\mathfrak{sl}_n$, with extra basis element I. Since I is central,

$$\mathfrak{gl}_n \cong \mathbb{C} \oplus \mathfrak{sl}_n.$$

The *center* of a Lie algebra $\mathfrak{g}$ is the maximal subspace $Z \subseteq \mathfrak{g}$ such that $[Z, \mathfrak{g}] = 0$.

(3) For $V = \mathbb{C}^n$, let $\langle , \rangle : V \otimes V \to \mathbb{C}$ be a *symmetric* form on V, i.e. $\langle u, v\rangle = \langle v, u\rangle$. Then

$$\mathfrak{so}_n(\mathbb{C}) = \mathfrak{so}(V) = \{x \in \mathfrak{sl}(V) \mid \langle xu, v\rangle + \langle u, xv\rangle = 0 \text{ for all } u, v \in V\}.$$

This is related to $\mathrm{SO}(V) = \{X \in \mathrm{SL}(V) \mid \langle Xu, Xv\rangle = \langle u, v\rangle \text{ for all } u, v \in V\}$, the *special orthogonal group.*

(4) For $V = \mathbb{C}^n$ with n even, let $\langle , \rangle : V \otimes V \to \mathbb{C}$ be a *skew symmetric* form on V, i.e. $\langle u, v\rangle = -\langle v, u\rangle$. Then

$$\mathfrak{sp}_n(\mathbb{C}) = \mathfrak{sp}(V) = \{x \in \mathfrak{sl}(V) \mid \langle xu, v\rangle + \langle u, xv\rangle = 0 \text{ for all } u, v \in V\}.$$

This is related to $\mathrm{Sp}(V) = \{X \in \mathrm{SL}(V) \mid \langle Xu, Xv\rangle = \langle u, v\rangle \text{ for all } u, v \in V\}$, the *symplectic group.*

(5) For $V = \mathbb{C}^n$, let $\langle , \rangle : V \times V \to \mathbb{C}$ be a *Hermitian* form on V, i.e. for vector spaces over $\mathbb{C}$,
 (i) $\langle u, c_1 v_1 + c_2 v_2\rangle = \bar{c_1}\langle u, v_1\rangle + \bar{c_2}\langle u, v_2\rangle$, and
 (ii) $\langle c_1 v_1 + c_2 v_2, u\rangle = c_1\langle v_1, u\rangle + c_2\langle v_2, u\rangle$.

Then

$$\mathfrak{su}(\mathbb{C}) = \mathfrak{su}(V) = \{x \in \mathfrak{sl}(V) \mid \langle xu, v\rangle + \overline{\langle u, xv\rangle} = 0 \text{ for all } u, v \in V\}.$$

This is related to $\mathrm{SU}(V) = \{X \in \mathrm{SL}(V) \mid \langle Xu, Xv\rangle = \langle u, v\rangle \text{for all } u, v \in V\}$, the *special unitary group*.

Notice that most of these algebras have the same structure, only with different types of forms. The families of algebras $\mathfrak{sl}(V)$, $\mathfrak{so}(V)$, and $\mathfrak{sp}(V)$ are the *classical Lie algebras*, and the above representations are their *standard* representations. They're named by type, as follow, and are almost always *simple* (meaning the only ideals of $\mathfrak{g}$ are 0 and itself, and $[\mathfrak{g}, \mathfrak{g}] \neq 0$):

Type A_r:	$\mathfrak{sl}_{r+1}(\mathbb{C})$	(distinct and simple for $r \geq 1$)
Type B_r:	$\mathfrak{so}_{2r+1}(\mathbb{C})$	(distinct and simple for $r \geq 2$)
Type C_r:	$\mathfrak{sp}_{2r}(\mathbb{C})$	(distinct and simple for $r \geq 3$)
Type D_r:	$\mathfrak{so}_{2r}(\mathbb{C})$	(distinct and simple for $r \geq 4$)

These are in fact all of the simple complex Lie algebras, except for the *exceptional Lie algebras*, E_6, E_7, E_8, F_4, and G_2.

See exercise 1: Calculate good bases for each of the classical types, and explore how some of the small types are redundant.

Matrix representations of Lie algebras. Sometimes you can deal with Lie algebras as concrete matrix Lie algebras, like we did above with. But to make general statements, Lie algebras are often treated as abstract algebraic structures defined with generators and relations, or using forms as above. Still, faithful (injective, structure-preserving) representations are integral in studying Lie algebras. Fortunately, we can always pass from an abstract Lie algebra to a faithful matrix representation via the *adjoint representation*:

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}) \qquad x \mapsto \mathrm{ad}_x = [x, \cdot], \quad \text{i.e. } \mathrm{ad}_x(y) = [x, y].$$

Just be careful–this is often not the same as the standard representation in general. For example, since $\mathfrak{sl}_2$ is 3-dimensional, $\mathrm{ad}(\mathfrak{sl}_2)$ is contained in $M_3(\mathbb{C})$, not $M_2(\mathbb{C})$. Also, this representation is faithful on simple Lie algebras, but if $\mathfrak{g}$ has a non-trivial center, additional tricks must be played to get a faithful representation.

2.2. **Categories, Functors, and the Universal Enveloping Algebra.** A *category* is a set of objects together with *morphisms* (functions) between them. Our favorite examples are

Alg = (algebras, algebra homomorphisms)
Lie = (Lie algebras, Lie algebra homomorphisms)

A *functor* is a map between categories

$$F : \mathcal{C}_1 \to \mathcal{C}_2$$

which associates to each object $X \in \mathcal{C}_1$ an object $F(X) \in \mathcal{C}_2$, associates to each morphism $f : X \to Y \in \mathcal{C}_1$ a morphism $F(f) : F(X) \to F(Y) \in \mathcal{C}_2$, and preserves both identity morphisms and composition of morphisms.

There is a functor

$$L : \mathrm{Alg} \to \mathrm{Lie}$$

where if $A \in \mathrm{Alg}$, then the underlying vector spaces of A and $L(A)$ are the same, but the product $a \cdot b \mapsto [a, b] = ab - ba$. There is also a functor

$$U : \mathrm{Lie} \to \mathrm{Alg}$$

where if $\mathfrak{g} \in$ Lie, then the underlying vector space of $U\mathfrak{g}$ is that of the algebra generated by the elements of $\mathfrak{g}$ with the relation $\underbrace{xy - yx}_{\text{as in } U\mathfrak{g}} = \underbrace{[x, y]}_{\text{as in } \mathfrak{g}}$. For example, if $[x, y] = 0$ in $\mathfrak{g}$ then $xy = yx$ in $U\mathfrak{g}$. $U\mathfrak{g}$ is said to be the *universal enveloping algebra* of $\mathfrak{g}$.

So now we have functors

$$L : \text{Alg} \to \text{Lie}$$

and

$$U : \text{Lie} \to \text{Alg}.$$

It may be tempting to speculate that these two functors are inverses of some kind. However, it is easy to see that $U\mathfrak{g}$ is rather large as compared to $\mathfrak{g}$ (often infinite dimensional), whereas $L(A)$ is not that much smaller than A (especially when A is finite dimensional). However, we do have the following theorem:

Theorem 2.1. *The functor U is* left-adjoint *to the functor L, i.e.,*

$$\text{Hom}_{\text{Alg}}(U\mathfrak{g}, A) \cong \text{Hom}_{\text{Lie}}(\mathfrak{g}, L(A))$$

as vector spaces.

3. Representations of $\mathfrak{g}$: a first try

A *representation* of $\mathfrak{g}$ is a representation of $U\mathfrak{g}$. A $U\mathfrak{g}$-*module* is a vector space M with a $U\mathfrak{g}$-action

$$U\mathfrak{g} \otimes M \to M,$$

where

$$(u, m) \mapsto um$$

which is bilinear, (i.e., if $c_1, c_2 \in \mathbb{C}$, then

$$(c_1u_1 + c_2u_2)m = c_1u_1m + c_2u_2m$$

$$u(c_1m_1 + c_2m_2) = c_1um_1 + c_2um_2$$

for $u_1, u_2 \in U\mathfrak{g}, m_1, m_2 \in M$) and

$$u_1(u_2m) = (u_1u_2)m.$$

Again, whenever we're using *tensor products*, we're just forcing bilinearity. Modules and representations are inseparable concepts–the module is the vector space and the representation tells $\mathfrak{g}$ how to act on that vector space.

3.1. **Dual spaces and Hopf algebras: lessons from Group theory.** Let G be a finite group and let $\mathfrak{g}$ be a finite-dimensional Lie algebra.

Trivial modules. The trivial module for a group G is

$$\mathbb{C}v \qquad \text{with action } gv = v \text{ for all } g \in G.$$

Analogously, the *trivial module* for a Lie algebra $\mathfrak{g}$ is

$$\mathbb{C}v \qquad \text{with action } xv = 0 \text{ for all } x \in \mathfrak{g}.$$

Tensor products. If M and N are G-modules, then $M \otimes N$ is a G-modules with action given by

$$g(m \otimes n) = gm \otimes gn \qquad \text{with } g \in G, m \in M, n \in N$$

and extended linearly. Similarly, if M and N are $\mathfrak{g}$-modules, the tensor product $M \otimes N$ is a $\mathfrak{g}$-module with action given by

$$x(m \otimes n) = xm \otimes n + m \otimes xn \qquad \text{with } x \in \mathfrak{g}, m \in M, n \in N,$$

and extended canonically, e.g.

$$xy(m \otimes n) = x(xm \otimes n + m \otimes xn) = \cdots .$$

Duals. The *dual* of a vectors space M is $M^* = \mathrm{Hom}(M, \mathbb{C})$. If M is a module over some algebra A, then M^* can also be an A-module, with specific actions from A. If $A = \mathbb{C}G$ is a group algebra, the action is given by

$$(g\varphi)(m) = \varphi(g^{-1}m) \qquad \text{with } g \in G, m \in M,$$

(check: $((g_1 g_2) \cdot \varphi)(m) = (g_1 \cdot (g_2 \cdot \varphi))(m) = (g_2 \cdot \varphi)(g_1^{-1}m) = \dots$), and extended linearly. For $A = U\mathfrak{g}$ an enveloping algebra, the action is analogously given by

$$(x\varphi)(m) = \varphi(-xm) \qquad \text{with } x \in \mathfrak{g}, m \in M,$$

and extended canonically, e.g.

$$((xy) \cdot \varphi)(m) = (x \cdot (y \cdot \varphi))(m) = (y \cdot \varphi)(-xm) = \varphi((yx)m).$$

Check:

$$((xy) \cdot \varphi)(m) = ((yx + [x, y]) \cdot \varphi)(m) = \varphi((xy - [x, y])m) = \varphi((yx)m).$$

There are two canonical morphisms

$$\begin{array}{rcl} \cup : M^* \otimes M & \to & \mathbb{C} \\ (\varphi, m) & \mapsto & \varphi(m) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \cap : \mathbb{C} & \to & M \otimes M^* \\ 1 & \mapsto & \sum_{b_i} b_i \otimes b_i^* \end{array}$$

where the sum is over any basis $\{b_i\}$ of M and $\{b_i^*\}$ is the dual basis in M^*. Usually, we think we need an inner product to produce dual bases. But in this context, we just need to calculate $\mathrm{Hom}(M, \mathbb{C})$. Notice that the second canonical map looks a lot like the trace, if you think of the basis vectors are column vectors with a 1 in the i^{th} place, and dual basis vector are row vectors with a 1 in the i^{th} place.

To summarize, what we've really done here is to define a Hopf algebra:

Definition. A *Hopf algebra* is an algebra U with three maps

$$\Delta : U \to U \otimes U,$$

$$\varepsilon : U \to \mathbb{C},$$

$$S : U \to U$$

such that

(1) If M and N are U-modules, then $M \otimes N$ with action

$$x(m \otimes n) = \sum_x x_{(1)}m \otimes x_{(2)}n$$

where $\Delta(x) = \sum_x x_{(1)} \otimes x_{(2)}$, is a U-module. [Note: this notation we're using is called *Sweedler notation*]

(2) The vector space $\mathbb{C} = v\mathbb{C}$, with actions $xv_1 = \varepsilon(x)v_1$ is a U-module.

(3) If M is a U-module then $M^* = \mathrm{Hom}(M, \mathbb{C})$ with action
$$(x\varphi)(m) = \varphi(S(x)m)$$
is a U-module.

(4) The canonical maps
$$\cup : M^* \otimes M \to \mathbb{C}$$
and
$$\cap : \mathbb{C} \to M \otimes M^*$$
are U-module homomorphisms.

Examples.

(1) If G is a group and $U = \mathbb{C}G = \mathbb{C}\{g \in G\}$ with
$\Delta(g) = g \otimes g$ the *coproduct*
$\varepsilon(g) = 1$ the *counit*
$S(g) = g^{-1}$ the *antipode*
is a Hopf algebra

(2) If $\mathfrak{g}$ is a Lie algebra, then $U = U\mathfrak{g}$ is a Hopf algebra with
$\Delta(x) = x \otimes 1 + 1 \otimes x$,
$\varepsilon(x) = 0$
$S(x) = -x$
for $x \in \mathfrak{g}$

For a very long time, these were the only two examples! But when I hear "Hopf algebra", I really hear "tensor products of modules are still modules".

3.2. **Representations of $\mathfrak{sl}_2(\mathbb{C})$.** Recall that $\mathfrak{sl}_2$ is the $\mathbb{C}$-span of elements x, y, h with bracket
$$[x, y] = h, \qquad [h, x] = 2x, \qquad \text{and} \qquad [h, y] = -2y.$$

Suppose M is an $\mathfrak{sl}_2$-module. So h acts on M, and morally, h is just a complex-valued matrix. So h has at least one eigenvalue and eigenvector. Let v be an eigenvector for h, i.e.
$$hv = \lambda v \qquad \text{for some } \lambda \in \mathbb{C}.$$
Further,
$$\begin{aligned} h(xv) &= (xh + [h, x])v \\ &= (x\lambda + 2x)v \\ &= (\lambda + 2)xv. \end{aligned}$$
Do it again:
$$\begin{aligned} h(x^2 v) &= (xh + [h, x])xv \\ &= (x(\lambda + 2) + 2x^2)v \\ &= (\lambda + 4)x^2 v. \end{aligned}$$
In general,
$$h(x^\ell)v = (\lambda + 2\ell)(x^\ell v).$$
In short, x generates more eigenvectors for h.

If M is finite dimensional, then $x^\ell v = 0$ for some $\ell \in \mathbb{Z}^+$. Let v^+ be maximally non-zero, i.e. v^+ satisfies
$$xv^+ = 0 \qquad \text{and} \qquad hv^+ = \mu v \qquad \text{for some } \mu \in \mathbb{C}^\times.$$

If M is simple, then v^+ generates M, i.e. $U\mathfrak{g}v^+ = M$. But h (basically) fixes v^+, and x kills v^+. What about y? Similarly to x,

$$h(yv^+) = (yh + [h, y])v^+ = (y\mu - 2y)v^+ = (\mu - 2)(yv^+)$$

and $h(y^2v^+) = (\mu - 4)(y^2v^+)$, and so on:

$$h(y^\ell v^+) = (\mu - 2\ell)(y^\ell v^+).$$

But again, M is finite-dimensional, $y^\ell v^+ = 0$ for some $\ell \in \mathbb{Z}^+$. Let $d \in \mathbb{Z}^+$ be such that $y^d v^+ \neq 0$ and $y^{d+1}v = 0$.

Back to x–what's the action of x on $\{v^+, yv^+, \dots, y^d v^+\}$?

$$\begin{aligned}
xv^+ &= 0\\
x(yv^+) &= (yx + [x, y])v^+ = (0 + h)v^+ = \mu v^+\\
x(y^2v^+) &= (yx + [x, y])yv^+ = y(\mu v^+) + h(yv^+) = (2\mu - 2)(yv^+)\\
x(y^3v^+) &= (yx + [x, y])y^2v^+ = y(2\mu - 2)yv^+ + h(y^2v^+) = (3\mu - 6)(y^2v^+)\\
x(y^4v^+) &= (yx + [x, y])y^3v^+ = y(3\mu - 6)y^2v^+ + h(y^3v^+) = (4\mu - 12)(y^3v^+)
\end{aligned}$$

and so on:

$$x(y^\ell v^+) = (\ell\mu - 2\binom{\ell}{2})y^{\ell-1}v^+ = (\ell\mu - \ell(\ell - 1))y^{\ell-1}v^+ = \ell(\mu - (\ell - 1))$$

So if M is simple, then $\{v^+, yv^+, \dots, y^d v^+\}$ is a basis of M, which is also a set of simultaneous eigenvectors for h.

$$\begin{tikzcd}
v^+ \arrow[r, bend left, "y"] & yv^+ \arrow[l, bend left, "x"] \arrow[r, bend left, "y"] & y^2v^+ \arrow[l, bend left, "x"] \arrow[r, bend left, "y"] & \cdots \arrow[l, bend left, "x"] \arrow[r, bend left, "y"] & y^{d-1}v^+ \arrow[l, bend left, "x"] \arrow[r, bend left, "y"] & y^d v^+ \arrow[l, bend left, "x"]
\end{tikzcd}$$

In summary, the $\mathfrak{sl}_2$-action is given by:

- h is a diagonal matrix with μ, $\mu - 2$, $\mu - 4$, $\dots$, $\mu - 2d$ on the diagonal,
- y has ones on the sub-diagonal and zeros elsewhere, and
- x has the weights μ, $2\mu - 2$, $3\mu - 6$, $\dots$, $d(\mu - (d - 1))$ on the super-diagonal.

$$h = \begin{pmatrix} \mu &&&& \\ & \mu - 2 &&& \\ && \mu - 4 && \\ &&& \ddots & \\ &&&& \mu - 2d \end{pmatrix} \quad y = \begin{pmatrix} 0 &&&& \\ 1 & 0 &&& \\ & 1 & 0 && \\ &&& \ddots & \\ &&& 1 & 0 \end{pmatrix} \quad x = \begin{pmatrix} 0 & \mu &&& \\ & 0 & 2\mu - 2 && \\ && 0 & 3\mu - 6 & \\ &&& \ddots & d(\mu - (d - 1)) \\ &&&& 0 \end{pmatrix}$$

But $h = [x, y] = xy - yx$, so if we multiply these matrices together, we get the relation that

$$\mu - 2d = 0 - d(\mu - (d - 1))$$

which is the same as

$$(d + 1)\mu = d(d - 1 + 2) = d(d + 1),$$

and so

$$\mu = d = \dim(M) - 1.$$

Theorem 3.1. *The irreducible finite dimensional $\mathfrak{sl}_2$ modules $L(d)$ are indexed by $d \in \mathbb{Z}_{\geq 0}$ with basis $\{v^+, yv^+, y^2v^+, \ldots, y^dv^+\}$ and action*

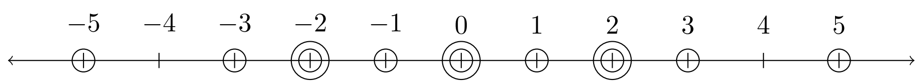
$$h(y^\ell v^+) = (d-2\ell)(y^\ell v^+), \quad x(y^\ell v^+) = \ell(d+1-\ell)(y^{\ell-1}v^+), \quad y(y^\ell v^+) = y^{\ell+1}v^+.$$

Suppose you've got some mystery finite-dimensional $\mathfrak{sl}_2$-module, M. How do we know what module we're talking about? Since h is diagonalizable on each irreducible component, M decomposes as a $\mathbb{C}h$-module into one-dimensional summands. Namely, it has a decomposition into *weight spaces* (otherwise known as eigenspaces):

$$M = \bigoplus_{\mu \in \mathbb{C}} M_\mu \qquad \text{where} \qquad M_\mu = \{m \in M \mid hm = \mu m\}.$$

We call M_μ the μ-weight space. We know the irreducible summands have weight spaces that are symmetric and have the same parity. So we can unpack them one at a time, starting with the longest strings. For example, if the weight spaces of M look like

μ	$i < -5$	-5	-4	-3	-2	-1	0	1	2	3	4	5	$5 < i$
$\dim(M_\mu)$	0	1	0	1	2	1	2	1	2	1	0	1	0



then $M \cong L(5) \oplus 2L(2)$.

Let M and N be finite-dimensional $\mathfrak{sl}_2$-modules. Again, Theorem 3.1 tells us that we can pick weight bases for M and N,

$$M = \mathbb{C}\{m_1, \ldots, m_r\} \qquad \text{and } N = \mathbb{C}\{n_1, \ldots, n_s\},$$

with

$$hm_i = h(m_i)m_i \quad \text{and} \quad hn_j = h(n_j)n_j \qquad \text{with } h(m_i), h(n_j) \in \mathbb{C}.$$

So

$$h \cdot (m_i \otimes n_j) = hm_i \otimes n_j + m_i \otimes hn_j = (h(m_i) + h(n_j))(m_i \otimes n_j).$$

So $M \otimes N$ has weight basis

$$\{m_i \otimes n_j \mid i = 1, \ldots, r, j = 1, \ldots, s\}.$$

Further, if the weight space decompositions of M and N are

$$M = \bigoplus_\alpha M_\alpha \quad \text{and} \quad N = \bigoplus_\beta N_\beta,$$

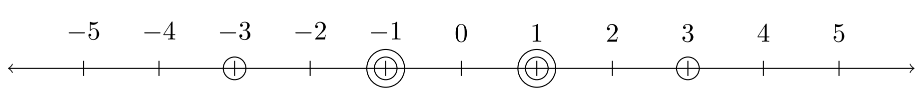
then the weight space decomposition of $M \otimes N$ is

$$M \otimes N = \bigoplus_{\alpha,\beta} (M_\alpha \otimes N_\beta), \qquad \text{where} \quad (M \otimes N)_\gamma = \bigoplus_{\substack{\alpha,\beta \\ \alpha+\beta=\gamma}} (M_\alpha \otimes N_\beta). \tag{3.1}$$

For example, since the weight spaces of $M = L(2)$ and $N = L(1)$ are given by

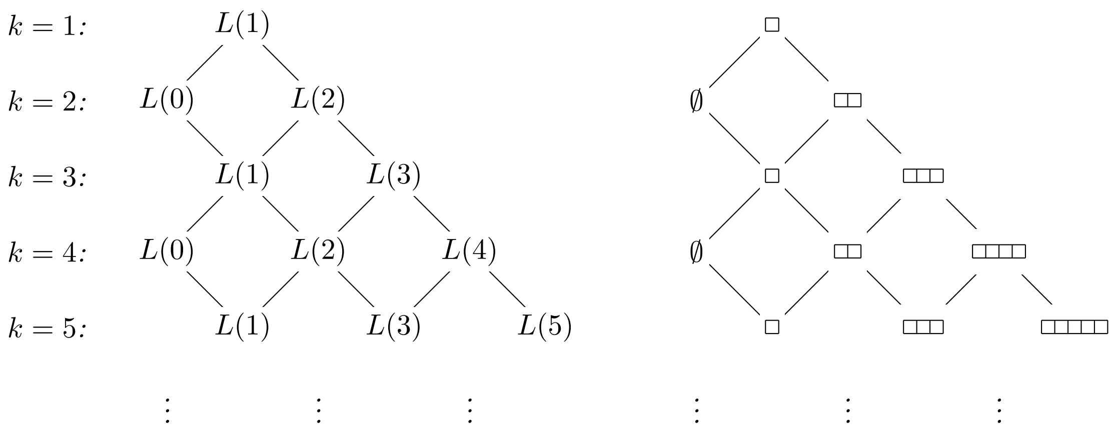
$$M = M_{-2} \oplus M_0 \oplus M_2 \qquad \text{and} \qquad N = N_{-1} \oplus N_1,$$

the weight space decomposition of $L(2) \otimes L(1)$ is



and so $L(2) \otimes L(1) \cong L(3) \oplus L(1)$. For a general decomposition, see homework.

Example. *For any $d > 0$, $L(d) \otimes L(1) = L(d+1) \oplus L(d-1)$. So the dimension of $L(a)$ in $L(1)^{\otimes k}$ is given by the number of downward-moving paths from $L(1)$ on level on, to $L(a)$ on level k in the lattice*



See exercise 2: Play around with some $\mathfrak{sl}_2$-modules.

Where we're headed: This characterization was pretty nice. It makes us feel like $\mathfrak{sl}_2$ was special, and made characterizing its representation theory easy. But actually, there's a much broader class of Lie algebras (finite dimensional complex semisimple Lie algebras) which inherits its representation theory all from $\mathfrak{sl}_2$. We'll see that this class has lots of $\mathfrak{sl}_2$'s as subalgebras, and how we can construct their representation theory by pasting strings of weight spaces together.

Remark 3.2. If we remove the requirement that M be finite-dimensional, a couple of things can happen.

(1) Suppose M has a non-zero element v^+ with $xv^+ = 0$ and $hv^+ = \mu v^+$ for some $\mu \in \mathbb{C}$ (we call v^+ *primitive*). Then M is simple if and only if it is generated by v^+, and we still call v^+ a highest weight vector. But we showed that M is finite-dimensional if and only if $\mu \in \mathbb{Z}_{\geq 0}$. So there are lots of simple highest weight modules that are not finite-dimensional. These still have reasonable tensor product rules and are reasonable to identify.

(2) If M doesn't have a primitive element but h still has an eigenvector v with weight λ, then $\{v, y^\ell v, x^\ell v \mid \ell \in \mathbb{Z}_0\}$ form a weight basis with weights amongst $\lambda + \mathbb{Z}$. These are trickier.

4. Finite dimensional complex semisimple Lie algebras

The most common phrase to all of my research is

"Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra."

It's a powerful phrase, with a lot of content, so let's unwind. It means

($*$) $\mathfrak{g}$ is a finite dimensional vector space,
($*$) $\mathfrak{g}$ is a vector space over $\mathbb{C}$,
($*$) $\mathfrak{g}$ is a Lie algebra, and
($*$) $\mathfrak{g}$ is semisimple.

What's semisimple?

Definition 1: semisimple Lie algebras decompose into simple pieces. An *ideal* of $\mathfrak{g}$ is a subspace $\mathfrak{a}$ such that if $x \in \mathfrak{g}$, $a \in \mathfrak{a}$, then $[x, a] \in \mathfrak{a}$. A *simple* Lie algebra is a Lie algebra with no non-trivial proper ideals and $[\mathfrak{g}, \mathfrak{g}] \neq 0$. A Lie algebra $\mathfrak{g}$ is *semisimple* if it is a direct sum of simple Lie algebras,

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_\ell$$

as Lie algebras. A Lie algebra $\mathfrak{g}$ is *reductive* if it is a direct sum of simple and abelian Lie algebras.
Definition 2: semisimple Lie algebras let their modules decompose into simple pieces. A $\mathfrak{g}$-module is *simple* if it has no non-trivial proper submodules. A $\mathfrak{g}$-module M is *semisimple* if M is a direct sum of simple $\mathfrak{g}$-modules:

$$M \cong M_1 \oplus \cdots \oplus M_\ell$$

as $\mathfrak{g}$-modules. A Lie algebra $\mathfrak{g}$ is *semisimple* if it has trivial center and all of the finite dimensional $\mathfrak{g}$-modules are semisimple.

Are these two the same? Well, first, $\mathfrak{g}$ is a $\mathfrak{g}$-module, so one is a special case of two. But in some sense, all $\mathfrak{g}$-modules are controlled by $\mathfrak{g}$, i.e. $\mathfrak{g}$ and $\{\mathfrak{g}\text{-modules}\}$ are the same data. There are *reconstruction theorems* explaining how to "reconstruct" $\mathfrak{g}$ only from information about the category of $\mathfrak{g}$-modules. So yes!

4.1. **Forms: symmetric, bilinear, invariant, and non-degenerate.** A *symmetric* bilinear form is a map

$$\langle , \rangle : M \otimes M \to \mathbb{C}$$

such that

$$\langle x, y \rangle = \langle y, x \rangle$$

for $x, y \in M$. Given such a form, notice that

$$M \to M^* \qquad \text{via} \qquad m \mapsto \langle m, \cdot \rangle$$

is a vector space homomorphism.

Let U be a Hopf algebra, and M be a U-module. An *invariant* symmetric bilinear form on M is a map $\langle , \rangle : M \times M \to \mathbb{C}$ such that

$$\langle x m_1, m_2 \rangle = \langle m_1, S(x) m_2 \rangle, \qquad \text{for } x \in U, m_1, m_2 \in M.$$

If G is a group, this means

$$\langle gm, n \rangle = \langle m, g^{-1} n \rangle \qquad \text{i.e. } \langle gm, gn \rangle = \langle m, n \rangle.$$

The invariant part means is that

$$M \to M^* \qquad \text{via} \qquad m \mapsto \langle m, \cdot \rangle$$

is a U-module homomorphism! On the level of vector spaces, the symmetric form doesn't know anything about multiplications in U, but an invariant form does.

Now let $U = U\mathfrak{g}$. Notice $\mathfrak{g}$ is a $\mathfrak{g}$-module under the adjoint action:

$$x \cdot y = \mathrm{ad}_x(y) = [x, y].$$

An invariant form for a Lie algebra is an *ad-invariant* form, meaning $\langle , \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ satisfies

$$\langle \mathrm{ad}_x(y), z \rangle = -\langle y, \mathrm{ad}_x(z) \rangle \qquad \text{i.e.} \qquad \langle [x, y], z \rangle = -\langle y, [x, z] \rangle.$$

We say $\langle , \rangle$ is *nondegenerate* if for all $x \in \mathfrak{g}$,

$$\langle x, \mathfrak{g} \rangle \neq 0.$$

Notice that $\mathrm{rad}\langle,\rangle = \{x \in \mathfrak{g} \mid \langle x, \mathfrak{g}\rangle = 0\}$ is an ideal of $\mathfrak{g}$, so if $\mathfrak{g}$ is simple, $\mathrm{rad}\langle,\rangle$ is zero or the whole thing. So nondegenerate forms give us an isomorphism

$$\begin{aligned} \mathfrak{g} &\to \mathfrak{g}^* \\ x &\mapsto \langle x, \cdot\rangle. \end{aligned} \tag{4.1}$$

In particular, on each simple piece, the only endomorphisms are constant multiples of the identity, so they're almost unique.

Every finite-dimensional semisimple complex Lie algebra has a canonical nondegenerate invariant bilinear symmetric (*NIBS*) form, the *Killing form*, given by

$$\langle x, y\rangle = \mathrm{Tr}(\mathrm{ad}_x \mathrm{ad}_y).$$

Since (4.1) is an isomorphism, any other NIBS form is a constant multiple of the Killing form. Other forms can be gotten by taking a faithful representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ and setting

$$\langle x, y\rangle = \mathrm{Tr}(\rho(x)\rho(y)).$$

A convenient choice is often the standard representation.

What if $\mathfrak{g}$ isn't semisimple? Reductive is actually good enough. In general, take $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{a}$ where $\mathfrak{g}_0$ is semisimple and $\mathfrak{a}$ is abelian. Let ρ be the adjoint representation on $\mathfrak{g}_0$ and the faithful diagonal representation on $\mathfrak{a}$; then the trace form on ρ is NIBS. Often there's something more computationally easy though.

Example. *Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}) = \{n \times n$ matrices with entries in $\mathbb{C}\}$ with bracket*

$$[x, y] = xy - yx.$$

Unfortunately, $\mathfrak{gl}_n$ is not simple or semisimple (it has a non-trivial center), but it is reductive, *meaning that it is the sum of a semisimple Lie algebra and an abelian Lie algebra. The Killing form is degenerate, but we can build another symmetric invariant bilinear form which is nondegenerate as follows. One basis of $\mathfrak{gl}_n$ is $\{E_{ij} \mid 1 \le i, j, \le n\}$ where E_{ij} has a 1 in the (i, j) entry and 0's elsewhere. This gives $\mathfrak{gl}_n$ a natural action on $V = \mathbb{C}\{v_1, \ldots, v_n\}$. So define $\langle,\rangle : \mathfrak{gl}_n \times \mathfrak{gl}_n \to \mathbb{C}$ by*

$$\langle x, y\rangle = \mathrm{Tr}(x_V y_V)$$

where x_V is the matrix of x acting on V. The dual basis with respect to $\langle\rangle$ is $E_{ij}^* = E_{ji}$. $U\mathfrak{gl}_n$ is generated by the symbols E_{ij}. So $E_{ij}E_{ij} \neq 0$ in $U\mathfrak{gl}_n$.

4.2. **Jordan-Chevalley decomposition, and lots of $\mathfrak{sl}_2$'s.** Way back when we looked at the representations of $\mathfrak{sl}_2$, I claimed that we would be able to past together strings of $\mathfrak{sl}_2$-representations to build the representation theory of other semisimple complex Lie algebras. That's because they contain a bunch of copies of $\mathfrak{sl}_2$! How?

Suppose V is a finite-dimensional vector space. If F is algebraically closed, every endomorphism $x \in \mathrm{End}(V)$ can be put into *Jordan canonical form*: the matrix form (which amounts to a good choice of basis) which consists of blocks corresponding to the eigenvalues λ of x, with λ's along the diagonal and 1's on the super-diagonal. So each block looks like

$$\begin{pmatrix} \lambda & 1 & & & 0 \\ & \lambda & 1 & & \\ & & \ddots & & \\ & & & & 1 \\ 0 & & & & \lambda \end{pmatrix} = \underbrace{\lambda I}_{s} + \underbrace{\begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & & \\ & & & & 1 \\ 0 & & & & 0 \end{pmatrix}}_{n}$$

The first summand s is *semisimple*, meaning the same thing as diagonalizable, i.e. that the roots of its minimal polynomial are distinct. The second summand n is *nilpotent*, meaning $n^\ell = 0$ for some $\ell \in \mathbb{Z}_{\geq 0}$. Moreover, s is central, meaning that $ns = sn$. Now, block-by-block, perform this decomposition, so that

$$x = x_s + x_n \qquad \text{where} \qquad \begin{array}{l} x_s \text{ consists of eigenvalues on the diagonal and 0's elsewhere, and} \\ x_n \text{ consists of the 1's above the diagonal and 0's elsewhere.} \end{array}$$

We call x_s the *semisimple part* (since it's semisimple) and x_n the *nilpotent part* (since x_n is indeed nilpotent). Since x_s and x_n commute block-by-block, $x_s x_n = x_n x_s$. Note that these properties (semisimple, nilpotent, and commuting) are not basis dependent–it's just that this is the easiest way to see what we call the (additive) *Jordan-Chevalley decomposition*:

For every $x \in \mathrm{End}(V)$, then there exist unique x_s semisimple and x_n nilpotent, with $\quad x = x_s + x_n \quad$ and $\quad [x_s, x_n] = 0.$

(see [Hum, §4.2] or [Ser, §I.5]). It can further be shown that any endomorphism commuting with x also commutes with x_s and x_n.

In a Lie algebra, an element x is *nilpotent* (resp. *semisimple*) if ad_x is nilpotent (resp. semisimple).

Theorem 4.1 (Jasobson-Morozov). *If x is a nilpotent element of a finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$, then there exist nilpotent y and semisimple h in $\mathfrak{g}$ such that*

$$[x, y] = h, \qquad [h, x] = 2x, \qquad [h, y] = -2y.$$

This choice is relatively unique (with some changes in constants). We call $\{x, y, h\}$ an $\mathfrak{sl}_2$ triple*.*

Proof. (sketch) Proof starts with using the Killing form to show the existence of an h, then uses Jordan-Chevalley to get a semisimple h. That semisimple h breaks $\mathfrak{g}$ into weight spaces, and derives a y that completes the triple.

Later, we'll see how to produce these more concretely. □

4.3. **Cartan subalgebras and roots.** Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra. A maximal abelian subalgebra $\mathfrak{h}$ consisting of semisimple elements is called a *Cartan subalgebra*.

Example. *For example, $\mathfrak{sl}_n$ has basis*

$$\{x_{ij} = E_{ij}, y_{ij} = E_{ji}, h_\ell = E_{\ell\ell} - E_{\ell+1,\ell+1} \mid 1 \leq i < j \leq n, 1 \leq \ell \leq n-1\}.$$

Our favorite Cartan subalgebra is

$$\mathfrak{h} = \mathbb{C}\{h_\ell \mid \ell = 1, \dots, n-1\}.$$

Example. *With a little bit of Jordan canonical form analysis, you showed on the homework that each of the Lie algebras of type A_r, B_r, C_r, and D_r all have Cartan subalgebras of dimension r.*

Some facts about Cartans.

1. Cartan subalgebras are generated by taking a (nice) semisimple element h and taking

$$\mathfrak{h} = \{g \in \mathfrak{g} \mid \mathrm{ad}_h(g) = 0\}$$

2. Cartan subalgebras exist and are unique up to inner automorphisms.
3. The centralizer of $\mathfrak{h}$ is $\mathfrak{h}$.
4. All elements of $\mathfrak{h}$ are semisimple.
5. The restriction of the Killing form to $\mathfrak{h}$ is non-degenerate.

For a quick tour, see [Ser, Ch. III]. The second property tells us that the *rank* of a semisimple Lie algebra, defined by

$$\mathrm{rank}(\mathfrak{g}) = \dim(\mathfrak{h})$$

is well-defined.

The *weights* of a Cartan $\mathfrak{h}$ is the dual set

$$\mathfrak{h}^* = \{\mu : \mathfrak{h} \to \mathbb{C}\}.$$

As before, since $\mathfrak{h}$ consists of semisimple elements, any finite-dimensional $\mathfrak{g}$ module M decomposes into $\mathfrak{h}$ weight spaces

$$M = \bigoplus_{\mu} M_\mu \qquad \text{where} \qquad M_\mu = \{m \in M \mid hm = \mu(h)m\}.$$

In particular, $\mathfrak{h}$ acts on $\mathfrak{g}$ by the adjoint action, and with respect to this action, $\mathfrak{g}$ decomposes into weight spaces as

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha} \mathfrak{g}_\alpha\right) \qquad \text{where} \quad \mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid \mathrm{ad}_h(x) = \alpha(h)x\}.$$

The set of weights

$$R = \{\alpha \in \mathfrak{h}^* \mid \alpha \neq 0, g_\alpha \neq 0\}$$

is called the set of *roots* of $\mathfrak{g}$.

Example. *For example, with $\mathfrak{h}$ as above, and $i \neq j$,*

$$\begin{aligned}[h_\ell, E_{ij}] &= (E_{\ell\ell} - E_{\ell+1,\ell+1})E_{ij} - E_{ij}(E_{\ell\ell} - E_{\ell+1,\ell+1}) \\ &= (\delta_{\ell,i} - \delta_{\ell+1,i} - \delta_{\ell,j} + \delta_{\ell+1,j})E_{ij}\end{aligned}$$

So as a $\mathfrak{h}$-module,

$$\mathfrak{sl}_n \cong h \oplus \left(\bigoplus_{\substack{\alpha_{i,j} \\ i \neq j}} g_{\alpha_{i,j}}\right) = h \oplus \left(\bigoplus_{\substack{\alpha_{i,j} \\ i < j}} g_{\alpha_{i,j}} \oplus g_{-\alpha_{i,j}}\right)$$

where

$$\alpha_{i,j}(h_\ell) = \delta_{\ell,i} - \delta_{\ell+1,i} - \delta_{\ell,j} + \delta_{\ell+1,j} = -\alpha_{j,i}(h_\ell)$$

(extended linearly) and

$$\mathfrak{g}_{\alpha_{ij}} = \mathbb{C}\{x_{ij}\} \qquad \textit{and} \qquad \mathfrak{g}_{-\alpha_{ij}} = \mathbb{C}\{y_{ij}\}.$$

Define

$$\begin{aligned}\varepsilon_i : \mathfrak{h} &\to \mathbb{C} && \textit{for } i = 1, \dots, n \\ h &\mapsto \mathrm{tr}(E_{ii}h)\end{aligned}$$

(i.e. it picks the ith diagonal element). So

$$\varepsilon_i(h_\ell) = \delta_{i,\ell} - \delta_{i,\ell+1} \qquad \textit{and} \qquad \alpha_{ij} = \varepsilon_i - \varepsilon_j.$$

So the set of roots for $\mathfrak{sl}_n$ is

$$R = \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \leq i < j \leq n\}.$$

Let $\langle , \rangle$ be a symmetric invariant nondegenerate bilinear form on $\mathfrak{g}$. Then the map

$$\begin{array}{ccc} \mathfrak{h} & \longrightarrow & \mathfrak{h}^* \\ h & \mapsto & \langle h, \cdot \rangle \\ h_\mu & \mapsto & \mu \end{array} \qquad \text{is an isomorphism,} \tag{4.2}$$

where h_μ is the unique element of $\mathfrak{h}$ such that

$$\langle h_\mu, h \rangle = \mu(h) \qquad \text{for all } h \in \mathfrak{h}.$$

Abusing notation, define a form $\langle , \rangle : \mathfrak{h}^* \otimes \mathfrak{h}^* \to \mathbb{C}$ by

$$\langle \lambda, \mu \rangle = \langle h_\lambda, h_\mu \rangle.$$

Then $\langle , \rangle$ is also symmetric, bilinear, and non-degenerate on $\mathfrak{h}^*$, and

$$\langle \lambda, \mu \rangle = \mu(h_\lambda) = \lambda(h_\mu).$$

Example. *Let $\langle , \rangle$ be trace form on the standard (defining) representation, and*

$$h_{ij} = E_{ii} - E_{jj} \qquad \textit{and } \alpha_{ij} = \varepsilon_i - \varepsilon_j$$

as before. Then $h_{\alpha_{ij}} = h_{ij}$. Further, $i < j$, we have

$$\langle h_{ij}, h_{k\ell} \rangle = \begin{cases} 2 & i = k \textit{ and } j = \ell \\ 1 & i = l \textit{ or } j = \ell \\ -1 & i = \ell \textit{ or } j = k \\ 0 & \textit{otherwise.} \end{cases}$$

So

$$\langle \varepsilon_i - \varepsilon_j, \varepsilon_k - \varepsilon_\ell \rangle = \begin{cases} 2 & i = k \textit{ and } j = \ell \\ 1 & i = l \textit{ or } j = \ell \\ -1 & i = \ell \textit{ or } j = k \\ 0 & \textit{otherwise.} \end{cases}$$

This is concurrent with $\{\varepsilon_1, \dots, \varepsilon_n\}$ being an orthonormal basis for $\mathbb{C}^n$, a fact which we will exploit later.

You have to be careful about the choices you make, though, when drawing these correspondences. As you'll see on the homework, if $(,)$ is the Killing form, then

$$(a, b) = 2n\langle a, b \rangle.$$

This can affect choices in a couple of ways, but with ε_i as above, $\{\varepsilon_1, \dots, \varepsilon_n\}$ is not an orthonormal basis with respect to $(,)$.

4.3.1. *Some facts about roots.* Let $\langle , \rangle$ be a NIBS form on $\mathfrak{g}$.

1. The adjoint action of $\mathfrak{g}_\alpha$ sends $\mathfrak{g}_\beta$ to $\mathfrak{g}_{\alpha+\beta}$:

$$\text{for } x_\alpha \in \mathfrak{g}_\alpha, \qquad \mathrm{ad}_{x_\alpha} : \mathfrak{g}_\beta \to \mathfrak{g}_{\alpha+\beta}.$$

In particular, $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{h}$.

Proof. For $h \in \mathfrak{h}$, $x_\alpha \in \mathfrak{g}_\alpha$, $x_\beta \in \mathfrak{g}_\beta$, we have

$$\begin{aligned} [h, [x_\alpha, x_\beta]] &= -[x_\alpha, [x_\beta, h]] - [x_\beta, [h, x_\alpha]] \\ &= -[x_\alpha, -\beta(h)x_\beta] - [x_\beta, \alpha(h)x_\alpha] \\ &= (\alpha(h) + \beta(h))[x_\alpha, x_\beta] = (\alpha + \beta)(h)[x_\alpha, x_\beta]. \end{aligned}$$

□

2. If $x_\alpha \in g_\alpha$ with $\alpha \neq 0$, then x_α is nilpotent.

Proof. An element x of a Lie algebra is nilpotent if ad_x is nilpotent. But
$$\mathrm{ad}_{x_\alpha}^\ell : \mathfrak{g}_\beta \to \mathfrak{g}_{\beta+\ell\alpha}.$$
Since $\mathfrak{g}$ is finite-dimensional, then for all $\beta \in \mathfrak{h}^*$, $\mathfrak{g}_{\beta+\ell\alpha} = 0$ for some $\ell \in \mathbb{Z}_{>0}$. □

3. If $\alpha \neq -\beta$, then $\langle \mathfrak{g}_\alpha, \mathfrak{g}_\beta \rangle = 0$.

Proof. Let $h \in \mathfrak{h}$ satisfy $(\alpha+\beta)(h) \neq 0$. Then
$$\begin{aligned}\alpha(h)\langle x_\alpha, x_\beta\rangle &= \langle [h, x_\alpha], x_\beta\rangle \\ &= -\langle x_\alpha, [h, x_\beta]\rangle \\ &= -\beta(h)\langle x_\alpha, x_\beta\rangle.\end{aligned}$$
So $\langle x_\alpha, x_\beta\rangle = 0$. □

4. The set of roots R is symmetric, i.e. if $\alpha \in R$, then $-\alpha \in R$.

Proof. If $\alpha \in R$, but $\mathfrak{g}_{-\alpha} = 0$, then $\langle \mathfrak{g}_\alpha, \mathfrak{g}\rangle = 0$ which contradicts the nondegenerance of the Killing form. □

Since R is symmetric, we can make a choice of the *positive* and *negative* roots
$$R = R^+ \sqcup R^- \quad \text{where} \quad R^- = \{-\alpha \mid \alpha \in R^+\}$$
For example, a standard choice for $\mathfrak{sl}_n$ is
$$R^+ = \{\varepsilon_i - \varepsilon_j \mid i < j\}.$$

---

5. The set $\{h_\alpha \mid \alpha \in R\}$ spans $\mathfrak{h}$, and so R spans $\mathfrak{h}^*$.

Proof. If it doesn't, then there's some $h \in \mathfrak{h}$ such that $\alpha(h) = 0$ for all $\alpha \in R$. This means that $[h, \mathfrak{g}_\alpha] = 0$ for all $\alpha \in R$, and so $h \in Z(\mathfrak{g}) = 0$, which is a contradiction. □

This says that we can choose a basis B for $\mathfrak{h}^*$ from R, of size $r = \mathrm{rank}(\mathfrak{g})$. More so, once we've chosen R^+, we can choose a basis from R^+. We'll see further that we can make this choice such that every root in R^+ is a positive integral combination of elements of B. Once this is done, we call B a *base* for R , and the elements of B are called the *simple roots.*

For example, a standard choice for $\mathfrak{sl}_n$ is
$$B = \{\beta_i = \varepsilon_i - \varepsilon_{i+1} \mid i = 1, \ldots, n-1\}.$$

---

6. If $x_\alpha \in \mathfrak{g}_\alpha$ and $y_\alpha \in \mathfrak{g}_{-\alpha}$ then $[x_\alpha, y_\alpha] = \langle x_\alpha, y_\alpha\rangle h_\alpha$ with h_α as in (4.2). So $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C}h_\alpha$.

Proof. For any $h \in \mathfrak{h}$,
$$\begin{aligned}\langle h, [x_\alpha, y_\alpha]\rangle &= \langle [h, x_\alpha], y_\alpha\rangle = \alpha(h)\langle x_\alpha, y_\alpha\rangle \\ &= \langle h, h_\alpha\rangle\langle x_\alpha, y_\alpha\rangle = \langle h, \langle x_\alpha, y_\alpha\rangle h_\alpha\rangle,\end{aligned}$$
showing
$$\langle \mathfrak{h}, [x_\alpha, y_\alpha] - \langle x_\alpha, y_\alpha\rangle h_\alpha\rangle = 0.$$
□

7. For all $\alpha \in R$, $\langle h_\alpha, h_\alpha \rangle \neq 0$.

Proof. (Show if $\langle h_\alpha, h_\alpha \rangle = 0$, then $h_\alpha \in Z(\mathfrak{g})$, which is a contradiction.) $\square$

8. Every non-zero $x_\alpha \in \mathfrak{g}_\alpha$ is part of an $\mathfrak{sl}_2$-triple,

$$\mathfrak{s}_\alpha = \langle x_\alpha, y_\alpha, h_{\alpha^\vee} \rangle$$

with

$$y_\alpha \in \mathfrak{g}_{-\alpha} \qquad \text{and} \qquad h_{\alpha^\vee} = \frac{2h_\alpha}{\langle h_\alpha, h_\alpha \rangle}.$$

Proof. Since $\langle h_\alpha, h_\alpha \rangle \neq 0$, it's possible to choose a $y_\alpha \in g_{-\alpha}$ such that

$$\langle x_\alpha, y_\alpha \rangle = \frac{2}{\langle h_\alpha, h_\alpha \rangle} \qquad \text{so that} \qquad [x_\alpha, y_\alpha] = \frac{2h_\alpha}{\langle h_\alpha, h_\alpha \rangle} = h_{\alpha^\vee}.$$

Moreover,

$$[h_{\alpha^\vee}, x_\alpha] = \frac{2}{\langle h_\alpha, h_\alpha \rangle}[h_\alpha, x_\alpha] = 2\frac{\alpha(h_\alpha)}{\alpha(h_\alpha)}x_\alpha = 2x_\alpha.$$

Similarly, $[h_{\alpha^\vee}, y_\alpha] = -2y_\alpha$. $\square$

9. If $\alpha \in R$ and $c\alpha \in R$, then $c = \pm 1$.

Proof. Consider the $\mathfrak{s}_\alpha$-module V spanned by $\mathfrak{h}$ and $\{\mathfrak{g}_{c\alpha} \mid c \in \mathbb{C}^\times\}$. The weights of $h_{\alpha^\vee}$ on V are given by

$$0 \qquad \text{and} \qquad c\alpha(h_\alpha) = c\frac{2}{\alpha(h_\alpha)}\alpha(h_\alpha) = 2c$$

(for non-zero c such that $\mathfrak{g}_{c\alpha} \neq 0$). But $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2$, so these weights are all *integers*! So the only nonempty weight spaces $\mathfrak{g}_{c\alpha}$ are where $c \in \frac{1}{2}\mathbb{Z}$.

Now since the image of $\alpha : \mathfrak{h} \to \mathbb{C}$ is one-dimensional, its kernel has co-dimension 1, so is equal to the orthogonal complement of $\mathbb{C}h_\alpha$–and most importantly, $\mathfrak{s}_\alpha$ acts trivially on this kernel. Next, $\mathfrak{s}_\alpha$ acts simply on itself. So the only subfactors of V with even $\mathfrak{s}_\alpha$ weights are $\mathfrak{s}_\alpha$ itself, and the one-dimensional modules in $\ker(\alpha)$. In particular, since $h_{\alpha^\vee}$ acts on $\mathfrak{g}_\alpha$ by 2, $\mathfrak{g}_{c\alpha} = 0$ for all $c \in \mathbb{Z}_{>1}$. Further, if $\frac{1}{2}\alpha \in R$, this implies that $\alpha \notin R$, which is a contradiction. $\square$

10. For $\alpha \neq 0$, $\mathfrak{g}_\alpha$ is one-dimensional.

Proof. A direct consequence of the previous proof is that as vector spaces, s_α is spanned by $\mathbb{C}h_\alpha$, $\mathfrak{g}_\alpha$, and $\mathfrak{g}_{-\alpha}$. $\square$

So far, we've learned that all semisimple finite-dimensional Lie algebras $\mathfrak{g}$ look a whole lot like $\mathfrak{sl}_n$. Namely, they admit a *triangular decomposition* of $\mathfrak{g}$, given by

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \qquad \text{where } \mathfrak{n}^\pm = \bigoplus_{\alpha \in R^\pm} \mathfrak{g}_\alpha.$$

Notice that while they are not ideals,

$$\mathfrak{n}^-, \quad \mathfrak{h}, \quad \mathfrak{n}^+, \quad \text{and} \quad \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$$

are all subalgebras ($\mathfrak{b}$ is called the *Borel* subalgebra). Here,

$$\begin{aligned} \mathfrak{n}^- &= \text{ behave like strictly lower triangular matrices,} \\ \mathfrak{h} &= \text{ behave like diagonal matrices,} \\ \mathfrak{n}^- &= \text{ behave like strictly upper triangular matrices.} \end{aligned}$$

On a basis of $\mathfrak{g}$ which respects the triangular decomposition decomposition, $\mathfrak{h}$ are diagonal matrices with diagonal entries

$$\begin{cases} 0 & \text{on basis elements in } \mathfrak{h}\text{, and} \\ \alpha(h) & \text{on basis elements in } \mathfrak{g}_\alpha. \end{cases}$$

In particular, since $\mathfrak{g}_\alpha$ is one-dimensional, if $(,)$ is the Killing form, then for any $a, b \in \mathfrak{h}$,

$$(a, b) = \sum_{\alpha \in R} \alpha(a)\alpha(b) \tag{4.3}$$

Also, a triangular decomposition of $\mathfrak{g}$ also gives the enveloping algebra a triangular decomposition

$$U\mathfrak{g} = U^- \otimes U^0 \otimes U^+ \qquad \text{with} \qquad U^\pm = U\mathfrak{n}^\pm \text{ and } U^0 = U\mathfrak{h}.$$

The *Poincaré-Birkhoff-Witt* theorem tells us that if I put an ordering on R^+, there are bases

$$\left\{ \prod_{\alpha \in R^+} y_\alpha^{m_\alpha} \;\middle|\; y_\alpha \in \mathfrak{g}_{-\alpha}, m_\alpha \in \mathbb{Z}_{\geq 0} \right\} \quad \text{of } U^-, \tag{4.4}$$

$$\left\{ \prod_{\beta \in B} h_\beta^{m_\beta} \;\middle|\; m_\beta \in \mathbb{Z}_{\geq 0} \right\} \quad \text{of } U^0, \text{ and} \tag{4.5}$$

$$\left\{ \prod_{\alpha \in R^+} x_\alpha^{m_\alpha} \;\middle|\; x_\alpha \in \mathfrak{g}_{\alpha}, m_\alpha \in \mathbb{Z}_{\geq 0} \right\} \quad \text{of } U^+. \tag{4.6}$$

Therefore, $U\mathfrak{g}$ has basis consisting of elements

$$y_{\alpha_1}^{m_1} \cdots y_{\alpha_\ell}^{m_\ell} h_{\beta_1}^{m_1'} \cdots h_{\beta_r}^{m_r'} x_{\alpha_1}^{m_1''} \cdots y_{\alpha_\ell}^{m_\ell''}$$

where $R^+ = \{\alpha_1, \ldots, \alpha_\ell\}$ and $B = \{\beta_1, \ldots, \beta_r\}$.

---

11. For $\alpha, \beta \in R$,
(a) $\beta(h_{\alpha^\vee}) \in \mathbb{Z}$,
(b) $\beta - \beta(h_{\alpha^\vee})\alpha \in R$, and
(c) if $\beta \neq \pm\alpha$, and a and b are the largest non-negative integers such that

$$\beta - a\alpha \in R \qquad \text{and } \beta + b\alpha \in R,$$

then $\beta + i\alpha \in R$ for all $-a \leq i \leq b$ and $\beta(h_{\alpha^\vee}) = a - b$.

Proof. These are consequences of $V = \sum_i \mathfrak{g}_{\beta + i\alpha}$ being an irreducible $\mathfrak{s}_\alpha$-module. See homework. □

This last bit says that there's symmetry in strings of roots. Rewriting these properties in terms of just roots,

$$\beta(h_{\alpha^\vee}) = 2\frac{\langle \alpha, \beta\rangle}{\langle \alpha, \alpha\rangle} \qquad \text{so} \qquad \beta - 2\frac{\langle \alpha, \beta\rangle}{\langle \alpha, \alpha\rangle}\alpha \in R.$$

Further, if s_α is defined on the operator on R that acts by

$$s_\alpha(\beta) = \beta - 2\frac{\langle \alpha, \beta\rangle}{\langle \alpha, \alpha\rangle}\alpha, \tag{4.7}$$

then since

$$\left\langle \alpha, \beta - 2\frac{\langle \alpha, \beta\rangle}{\langle \alpha, \alpha\rangle}\alpha \right\rangle = \langle \alpha, \beta\rangle - 2\frac{\langle \alpha, \beta\rangle}{\langle \alpha, \alpha\rangle}\langle \alpha, \alpha\rangle = -\langle \alpha, \beta\rangle$$

$$\begin{aligned} s_\alpha(\beta)^2(\beta) &= s_\alpha\left(\beta - 2\frac{\langle \alpha, \beta\rangle}{\langle \alpha, \alpha\rangle}\alpha\right) \\ &= \beta - 2\frac{\langle \alpha, \beta\rangle}{\langle \alpha, \alpha\rangle}\alpha + 2\frac{\langle \alpha, \beta\rangle}{\langle \alpha, \alpha\rangle} = \beta. \end{aligned}$$

So $s_\alpha^2 = 1$. Geometrically, s_α is a reflection across the hyperplane in $\mathfrak{h}^*$ given by $\mathfrak{h}_\alpha = \{\lambda \in \mathfrak{h}^* \mid \langle \alpha, \lambda\rangle = 0$.

12. (Rationality) Let $B \subseteq R$ is a basis for R, and assume $\langle , \rangle$ is a positive rational multiple of the Killing form on each simple piece of $\mathfrak{g}$.

(a) $R \subseteq \mathbb{Q}B$.

(b) For any $\alpha, \beta \in R$, $\langle \alpha, \beta\rangle \in \mathbb{Q}$.

(c) The restriction of $\langle , \rangle$ to

$$\mathfrak{h}^*_{\mathbb{Q}} = \mathbb{Q}B \qquad \text{and} \qquad \mathfrak{h}^*_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Q}} \mathfrak{h}^*_{\mathbb{Q}}$$

is positive definite. Therefore $\mathfrak{h}^*_{\mathbb{Q}}$ and $\mathfrak{h}^*_{\mathbb{R}}$ (and thus $\mathfrak{h}_{\mathbb{Q}}$ and $\mathfrak{h}_{\mathbb{R}}$) are Euclidean spaces with inner product $\langle , \rangle$.

Proof. For (a), let $\alpha = \sum_{\beta \in B} c_\beta \beta \in R$. Since B is a basis, there is some $\gamma \in B$ for which $\alpha(h_{\gamma^\vee}) \neq 0$. With such a γ,

$$\alpha(h_{\gamma^\vee}) = 2\frac{\langle \alpha, \gamma\rangle}{\langle \gamma, \gamma\rangle} = \sum_{\beta \in B} c_\beta 2\frac{\langle \beta, \gamma\rangle}{\langle \gamma, \gamma\rangle} = \sum_{\beta \in B} c_\beta \beta(h_{\gamma^\vee}).$$

So since $\alpha(h_{\alpha'}) \in \mathbb{Z}$ for any $\alpha, \alpha' \in R$, this says the linear combination of integers with coefficients c_β is an integer. Since this is true for any α, γ, we have $c_\beta \in \mathbb{Q}$ for each β.

For (b), recall that every NIBS form is a constant multiple of the Killing form on each simple piece of $\mathfrak{g}$; let $r_\alpha \in \mathbb{Q}$ be such that (4.3) gives

$$\langle a, b\rangle = \sum_{\alpha \in R} r_\alpha \alpha(a)\alpha(b).$$

Then for any to $\lambda, \mu \in \mathfrak{h}^*$, we have

$$\langle \lambda, \mu\rangle = \langle h_\lambda, h_\mu\rangle = \sum_{\alpha \in R} r_\alpha \alpha(h_\lambda)\alpha(h_\mu) = \sum_{\alpha \in R} r_\alpha \langle \alpha, \lambda\rangle\langle \alpha, \mu\rangle.$$

In particular, by 11(a), for any $\beta \in R$,

$$\frac{1}{\langle \beta, \beta \rangle} = \frac{\langle \beta, \beta \rangle}{\langle \beta, \beta \rangle^2} = \sum_{\alpha \in R} r_\alpha \frac{\langle \alpha, \beta \rangle^2}{\langle \beta, \beta \rangle^2} = \sum_{\alpha \in R} \frac{r_\alpha}{4} \left(2 \frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \right)^2 \in \mathbb{Q}.$$

So $\langle \beta, \beta \rangle \in \mathbb{Q}$, and so $\langle \alpha, \beta \rangle \in \mathbb{Q}$.

For (c), for any $\lambda \in \mathfrak{h}^*_{\mathbb{Q}}$ (resp. $\mathfrak{h}_{\mathbb{R}}$), it follows similarly that $\langle \lambda, \lambda \rangle$ is the sum of squares of rational numbers (resp. real), and is therefore positive unless $\lambda = 0$. $\square$

This last part says that s_α acts not just on R, but on $\mathfrak{h}^*_{\mathbb{R}}$ by the reflection across the hyperplane $\mathfrak{h}_\alpha = \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle \lambda, \alpha \rangle = 0\}$. Note that $s_\alpha = s_{-\alpha}$ and $\mathfrak{h}_\alpha = \mathfrak{h}_{-\alpha}$, so we only need to consider reflections associated to positive roots. The group W generated by $\{s_\alpha \mid \alpha \in R^+\}$ is called the *Weyl group* associated to $\mathfrak{g}$.

Example. *Roots and weights of $\mathfrak{sl}_3$ Consider $\mathfrak{g} = \mathfrak{sl}_3$, and let $\langle , \rangle$ be the trace form on the standard representation. Then*

$$\mathfrak{h} = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -(a+b) \end{pmatrix} \right\} = \mathbb{C} \left\{ h_1 = h_{\beta_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, h_2 = h_{\beta_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}$$

with $\beta_1 = \varepsilon_1 - \varepsilon_2$ and $\beta_2 = \varepsilon_2 - \varepsilon_3$. The roots of $\mathfrak{sl}_3$ are

$$\{\varepsilon_i - \varepsilon_j \mid i \neq j\}$$

with

$$\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{C} E_{ij}.$$

Then we can choose $B = \{\beta_1, \beta_2\}$ to be the simple roots, and

$$R^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq 3\} = \{\beta_1, \beta_2, \beta_1 + \beta_2\},$$

so that R^+ has the nice feature that it's made up of positive integral combinations of elements of B.

The rest of the triangular decomposition is given by

$$\mathfrak{n}^+ = \sum_{\alpha \in R^+} \mathfrak{g}_\alpha, \qquad \text{where} \quad \mathfrak{g}_\alpha = \mathbb{C} x_\alpha \quad \text{with} \quad \begin{array}{c} x_{\beta_1} = E_{1,2} \\ x_{\beta_2} = E_{2,3} \\ x_{\beta_1+\beta_2} = E_{1,3} \end{array}$$

(check: $[x_{\beta_1}, x_{\beta_2}] = x_{\beta_1+\beta_2}$) and

$$\mathfrak{n}^- = \sum_{\alpha \in R^-} \mathfrak{g}_\alpha, \qquad \text{where} \quad \mathfrak{g}_\alpha = \mathbb{C} y_\alpha \quad \text{with} \quad y_\alpha = x_\alpha^T.$$

For each of $\alpha \in R^+$, notice that $\langle \alpha, \alpha \rangle = 2$, so that $h_\alpha = h_{\alpha^\vee}$. Thus, we can check that $\mathfrak{s}_\alpha = \langle x_\alpha, y_\alpha, h_\alpha \rangle$ is the $\mathfrak{sl}_2$ triple associated with α.

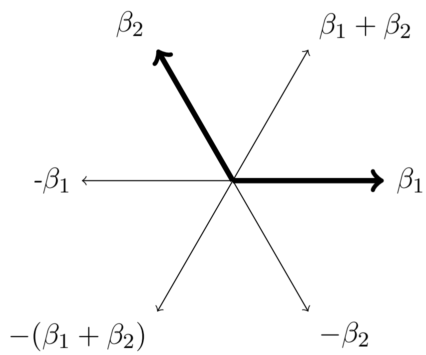
Although before we thought of $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ as an orthonormal basis for $\mathbb{R}^3$,

$$\mathfrak{h}_{\mathbb{R}} = \mathbb{R}\{\beta_1 = \varepsilon_1 - \varepsilon_2, \beta_2 = \varepsilon_2 - \varepsilon_3\} = \mathbb{R}^2$$

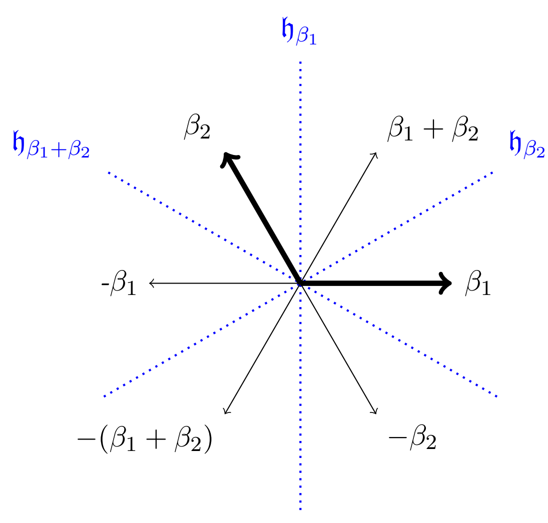
is two dimensional. In particular,

$$\beta_1 \angle \beta_2 = \arccos \frac{\langle \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3 \rangle}{\sqrt{\langle \varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_2 \rangle \langle \varepsilon_2 - \varepsilon_3, \varepsilon_2 - \varepsilon_3 \rangle}} = \arccos \frac{-1}{2} = \frac{2\pi}{3}.$$

So we can can plot R as

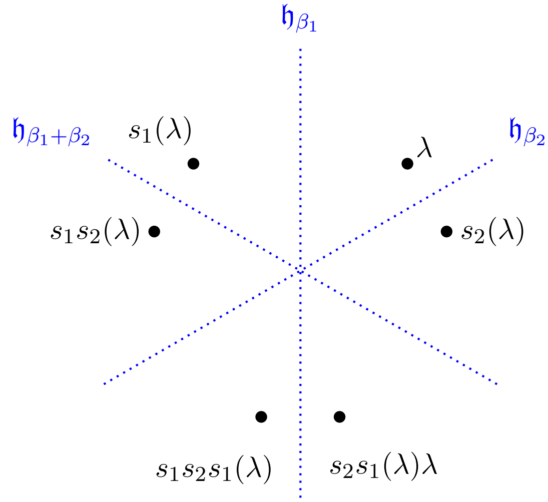


Throwing in the hyperplanes $\mathfrak{h}_{\beta_1}$, $\mathfrak{h}_{\beta_2}$, and $\mathfrak{h}_{\beta_1+\beta_2}$, we have



Notice that the positive roots α define a positive side of each hyperplane, given by the side that α sits on. We call the fundamental chamber *the set of points which lie on the positive side of every hyperplane $\mathfrak{h}_\alpha$.*

Let $s_1 = s_{\beta_1}$ and $s_2 = s_{\beta_2}$. Then for any $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$ (not laying on a hyperplane), the W-orbit of λ looks like



In particular, $W\alpha = R$ for any $\alpha \in R$. Also, it's now easy to see that

$$W = \langle s_\alpha \mid \alpha \in B\rangle = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1, s_1s_2s_1 = s_2s_1s_2\rangle \cong S_3.$$

In general, if s_α and s_β are reflections across hyperplanes that have an angle of $2\pi/3$ between them, they'll satisfy $s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta$. Reflection across perpendicular hyperplanes commute.

5. Highest weight representations

Fix a finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$ and a NIBS form $\langle , \rangle$.

Remark 5.1. I said before that we'll be able to find a really nice basis of roots for $\mathfrak{h}^*$. Namely, there exists a basis B comprised of linearly independent roots, such that every element of R is either a non-negative or a non-positive integral combination of elements of B. Any such basis is called a *base*. The elements of the base are called the *roots*simple roots. More, this says that we can choose R^+ so that for all $\alpha \in R^+$, $\alpha = \sum_{\beta \in B} z_\beta \beta$ such that $z_\beta \in \mathbb{Z}_{\geq 0}$. We're going to put off the proof of the existence of a base for just a little longer (see Proposition 6.2), and assume it for now.

Taking Remark 5.1 for granted, we can classify the finite-dimensional representation theory of $\mathfrak{g}$.
Since $\mathfrak{h}$ consists of pairwise commuting semisimple elements, their action is simultaneously diagonalizable on any representation. So just like the adjoint action of $\mathfrak{g}$ on itself, if V is a simple finite-dimensional $\mathfrak{g}$-module, V decomposes into weight spaces for $\mathfrak{h}$:

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda \qquad \text{where} \qquad V_\lambda = \{v \in V \mid hv = \lambda(h)v\}.$$

Further, for any $h \in \mathfrak{h}$, $v \in V_\lambda$ and $x \in \mathfrak{g}_\alpha \neq 0$, we have

$$hxv = (xh + [h,x])v = (\lambda(h) + \alpha(h))xv, \qquad \text{and so} \qquad hx^\ell v = (\lambda(h) + \ell\alpha(h))x^\ell v. \tag{5.1}$$

Similarly, for $\alpha_i \in R$ (not necessarily distinct) and $x_i \in \mathfrak{g}_{\alpha_i}$,

$$hx_1 \cdots x_m v = \left(\lambda(h) + \sum_{i=1}^m \alpha_i(h)\right) x_1 \cdots x_m v. \tag{5.2}$$

The difference between the general case and the case of $\mathfrak{sl}_2$ in Section 3.2 is that there are multiple positive root spaces for most $\mathfrak{g}$. So to force the existence of a highest weight vector, we need to find a basis of $\mathfrak{h}$ for which for every basis element h, $\sum_{i=1}^m \alpha_i(h)$ weakly increases as $x_1 \cdots x_m$ acquires more terms from positive root spaces. If B is a base for R as in Remark 5.1, then $\{h_\beta \mid \beta \in B\}$ is one basis of $\mathfrak{h}$. However, this will not do the trick in general–what we need is basis of $\mathfrak{h}^*$ such that

$$\text{for any } \alpha \in R^+,\ \langle \alpha, \omega \rangle \geq 0 \text{ for all basis elements } \omega \tag{5.3}$$

so that $\alpha(h_\omega) \geq 0$ for all basis elements h_ω of h.

Fix a base $B = \{\beta_1, \dots, \beta_r\}$ and $R^+ = R \cap \mathbb{Z}_{\geq 0} B$. Let $\Omega = \{\omega_1, \dots, \omega_r\}$ be a set of roots satisfying

$$\langle \omega_i, \beta_j \rangle = c_i \delta_{i,j} \text{ for } \beta_j \in B, \quad \text{with } c_i \in \mathbb{R}_{>0}.$$

Since $\langle , \rangle$ is NIBS, $\{\omega_1, \dots, \omega_r\}$ is unique up to choice of the c_i's. Then Remark 5.1 ensures (5.3).

Example. *In $\mathfrak{sl}_4$, if the simple roots are given by*

$$\beta_1 = \varepsilon_1 - \varepsilon_2, \qquad \beta_2 = \varepsilon_2 - \varepsilon_3, \qquad \beta_3 = \varepsilon_3 - \varepsilon_4,$$

then if $\nu = \frac{1}{4}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$, we have

$$\omega_1 = c_1(\varepsilon_1 + \nu), \qquad \omega_2 = c_2(\varepsilon_1 + \varepsilon_2 + 2\nu), \qquad \omega_3 = c_3(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \nu).$$

Now if $x_\alpha \in \mathfrak{g}_\alpha$ for $\alpha \in R^+$, $\omega \in \Omega$, and v_λ a weight vector of weight λ, the (5.2) tells us that x_α pushes the eigenvalue of h_ω on v_λ in the positive direction along the number line $\lambda(h_\omega) + \mathbb{R}$. Since V is finite-dimensional, only finitely many of the resulting weight vectors can be non-zero. So there exists some $v^+ \in V$, called the *highest weight vector*, which satisfies

$$hv^+ = \mu(h)v^+ \text{ for some } \mu \in \mathfrak{h}^* \qquad \text{and} \qquad xv^+ = 0 \text{ for all } x \in \mathfrak{n}^+.$$

If V is simple, then any non-zero element of V generates V, and so $V = U\mathfrak{g}v^+$.

Lemma 5.2. *Let V be a finite-dimensional simple $\mathfrak{g}$-module.*

(a) Then there is a highest weight vector $v^+ \in V$ satisfying

$$hv^+ = \mu(h)v^+ \textit{ for some } \mu \in \mathfrak{h}^*, \quad \mathfrak{n}^+ v^+ = 0, \quad \textit{and} \quad U\mathfrak{n}^- v^+ = V.$$

(b) V is spanned by weight vectors

$$\{y_{\alpha_1}^{m_1} \cdots y_{\alpha_\ell}^{m_\ell} v^+ \mid m_i \in \mathbb{Z}_{\geq 0}\} \quad \textit{with} \quad \begin{array}{l} R^+ = \{\alpha_1, \dots, \alpha_\ell\}, \textit{ and} \\ y_\alpha \in \mathfrak{g}_{-\alpha}, \end{array}$$

and

$$hyv^+ = \left(\mu - \sum_i m_i \alpha_i\right)(h)yv^+ \quad \textit{for} \quad y = y_{\alpha_1}^{m_1} \cdots y_{\alpha_\ell}^{m_\ell}.$$

(c) The weight spaces of V are

$$V_\lambda \qquad \textit{with } \lambda = \mu - \sum_{i=1}^r \ell_i \beta_i, \quad \ell_i \in \mathbb{Z}_{\geq 0},$$

where $B = \{\beta_1, \dots, \beta_r\}$ is a base for the roots of $\mathfrak{g}$. Further, $\dim(V_\mu) = 1$.

Proof. (a) We have already shown that every such V has a vector satisfying $\mathfrak{n}^+ v^+ = 0$ and $hv^+ = \mu(h)v^+$ for some $\mu \in \mathfrak{h}^*$. Recall that Poincaré-Birkhoff-Witt (4.4) says

$$Un^- = \mathbb{C}\{y_{\alpha_1}^{m_1} \cdots y_{\alpha_\ell}^{m_\ell} \mid m_i \in \mathbb{Z}_{\geq 0}\} \quad \text{with} \quad \begin{array}{l} R^+ = \{\alpha_1, \dots, \alpha_\ell\}, \text{ and} \\ y_\alpha \in \mathfrak{g}_{-\alpha}. \end{array}$$

So it remains to show that for any $x \in \mathfrak{n}^+$, $h \in \mathfrak{h}$, and y a monomial in $U\mathfrak{n}^-$,

(i) $hyv^+ = \nu(h)yv^+$ for some $\nu \in \mathfrak{h}^*$, and
(ii) $xyv^+ \in U\mathfrak{n}^- v^+$.

Both can be done inductively on the degree of y. So let $y = y_\alpha y'$ be a monomial in $U\mathfrak{n}^-$.

For (i), assume inductively that $y'v^+$ is a weight vector with weight λ. Then

$$h(yv^+) = (y_\alpha h + [h, y_\alpha])(y'v^+) = (\lambda(h)y_\alpha - \alpha(h)y_\alpha)y'v^+ = (\lambda - \alpha)(h)yv^+. \tag{5.4}$$

So each y_α pushes the weight of v^+ in the $-\alpha$ direction.

For (ii), assume inductively that $\mathfrak{n}^+ y'v^+ \subseteq U\mathfrak{n}^- v^+$, and consider

$$x_\beta(yv^+) = (x_\beta y_\alpha)(y'v^+) = y_\alpha x_\beta(y'v^+) + [x_\beta, y_\alpha](y'v^+).$$

By the induction hypothesis, if $x_\beta \in \mathfrak{g}_\beta$ for $\beta \in R^+$, then $x_\beta(y'v^+) \in U\mathfrak{n}^- v^+$. And since $[x_\beta, y_\alpha] \subseteq \mathfrak{g}_{\beta-\alpha}$, then either

$$\begin{aligned} \beta - \alpha \in R^- &\quad \text{so that } [x_\beta, y_\alpha] \in \mathfrak{n}^-, \\ \beta - \alpha = 0 &\quad \text{so that } [x_\beta, y_\alpha] \in \mathfrak{h} \text{ and so } [x_\beta, y_\alpha](y'v^+) = cy'v^+, \text{ or} \\ \beta - \alpha \in R^+ &\quad \text{so that } [x_\beta, y_\alpha] \in \mathfrak{n}^+. \end{aligned}$$

In all three cases, the desired result follows directly or inductively.

(b) Poincaré-Birkhoff-Witt and part (a) shows that $\{y_{\alpha_1}^{m_1} \cdots y_{\alpha_\ell}^{m_\ell} \mid m_i \in \mathbb{Z}_{\geq 0}\}$ forms a spanning set. The fact that these are weight vectors of the weight $(\mu - \sum_i m_i\alpha_i)$ follows inductively from (5.4).

(c) Since R^+ are all positive integral linear combinations over B, we have $\sum_j m_j\alpha_j = \sum_{i=1}^r \ell_i\beta_i$. So all of the weights of V differ from μ by integral linear combinations of the simple roots. Further, since $\sum_{i=1}^r \ell_i\beta_i = 0$ if and only if $y = 1$, $\dim(V_\mu) = 1$.

□

In general, we say an element v of a $\mathfrak{g}$-module V is a *primitive* element (of weight $\mu \in \mathfrak{h}^*$) if

$$hv = \mu(h)v \quad \text{and } \mathfrak{n}^+ v = 0.$$

For the following lemma, no assumptions about V being finite-dimensional or simple are made.

Lemma 5.3. *Let V be generated by primitive element v_μ of weight μ.*

(1) Parts (a)–(c) from Lemma 5.2 hold for V as well.
(2) V is indecomposable, and therefore simple.
(3) There is a unique (up to scaling) primitive element of V.
(4) Two modules $V^{(\mu)}$ and $V^{(\lambda)}$ generated by primitive elements v_μ and v_λ, respectively, are isomorphic if and only if $\mu = \lambda$.

Proof. (1) Everything in the proof following Lemma 5.2 depended only on the existence of a primitive element (whose existence was gleamed from V's irreducibility).

(2) From part (c) of Lemma 5.2, $\dim(V_\mu) = 1$. Now suppose $V = A \oplus B$ as an $\mathfrak{g}$-module. Then

$$1 = \dim(V_\mu) = \dim(A_\mu) + \dim(B_\mu).$$

So either $v_\mu \in A_\mu$, in which case $V = U\mathfrak{g}v \subseteq A \subseteq V$, or $v_\mu \in B$, in which case $V = U\mathfrak{g}v_\mu \subseteq B \subseteq V$. Both are contradictions.

(3) Let v_λ be a second primitive element of V with weight λ. By part (c) of Lemma 5.2,

$$\lambda = \mu - \sum_{i=1}^r \ell_i\beta_i \quad \text{and } \mu = \lambda - \sum_{i=1}^r n_i\beta_i,$$

with $\ell_i, n_i \geq 0$. So $\ell_i = -n_i$, which is only possible if $\ell_i = n_j = 0$. So $\mu = \lambda$. Further, $\dim(V_\mu) = 1$, so $v_\lambda = cv_\mu$.

(4) Part (3) tells us $\mu = \lambda$ is necessary. To show sufficiency, let A, B be generated by primitive v_a and v_b respectively, both of weight μ, and consider $V = A \oplus B$. Since $v_c = v_a + v_b \in V$ also has weight μ and is annihilated by $\mathfrak{n}^+$, it is also a primitive element of V of weight μ. Let C be the submodule generated by v_c. The projections

$$\pi_A : V \to A \quad \text{and} \quad \pi_B : V \to B$$

are both $\mathfrak{g}$-module homomorphisms. Since $\pi_A(v_c) = v_a$ and $\pi_B(v_c) = v_b$, so

$$A \subseteq \pi_A(C) \quad \text{and} \quad B \subseteq \pi_B(C).$$

But by part (2), C is simple, and so $A \cong \pi_A(C) \cong C \cong \pi_B(C) \cong B$.

□

So far, we know that every finite dimensional simple $\mathfrak{g}$-module is generated by a primitive element, and two such modules are isomorphic if and only if they are generated by primitive elements of the same weight. All that remains is to understand which weights appear as highest weights for $\mathfrak{g}$-modules (e.g. if $\mathfrak{g} = \mathfrak{sl}_2$, the highest weights are non-negative integers), and what the structure of the corresponding highest weight module is (e.g. if $\mathfrak{g} = \mathfrak{sl}_2$, the highest weight module of weight d is the $d+1$-dimensional module satisfying...).

To this end, we're going to think for a couple paragraphs about the ramifications of the presence of an isomorphic copy of $\mathfrak{sl}_2$ associated to each $\alpha \in R^+$:

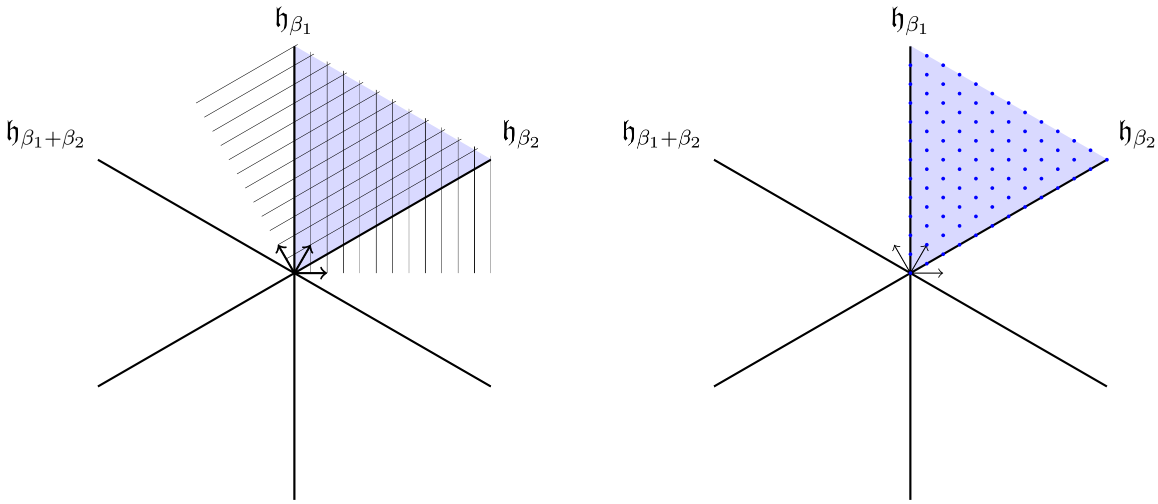
$$\mathfrak{s}_\alpha = \langle x_\alpha, y_\alpha, h_{\alpha^\vee} \mid x_\alpha \in \mathfrak{g}_\alpha, y_\alpha \in \mathfrak{g}_{-\alpha}, h_{\alpha^\vee} \in \mathfrak{h} \rangle.$$

The result of this rambling follows the example below, in Proposition 5.4.

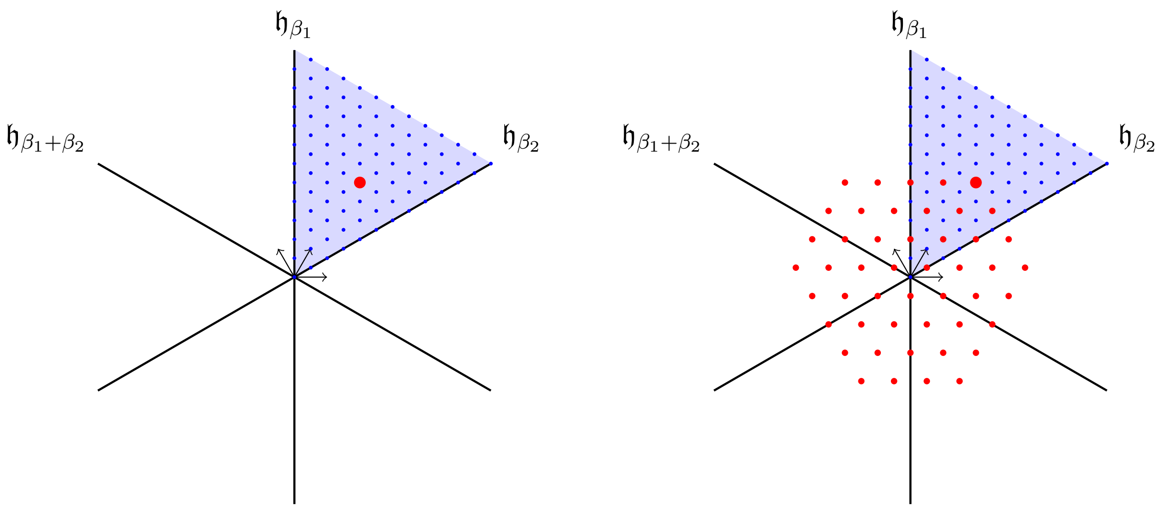
The action of $\mathfrak{s}_\alpha$ means that for each positive root α, not only does y_α push the weight of a weight vector one unit in the $-\alpha$ direction, but V restricts to an $\mathfrak{sl}_2$ module. This means that the α string is symmetric around the hyperplane $\mathfrak{h}_\alpha$, and x_α reverses each step made by y_α. (Aside: You might think "wait, I thought the weights of the $\mathfrak{sl}_2$-modules were integers!"–but it's ok since $\alpha(h_{\alpha^\vee}) = 2$.) So any weight of V is either an integer or half-integer multiple of α away from the hyperplane $\mathfrak{h}_\alpha$. Since every positive root is a positive integer linear combination of simple roots, we only need consider the simple roots.

More, if v^+ has weight μ, μ has to sit to the positive side of each hyperplane $\mathfrak{h}_{\beta_i}$ (the side that the positive root lies on), so that μ is in the closure of the fundamental chamber C. If μ is at a distance of $\ell_i||\beta_i||$ for some integer or half integer ℓ_i, this distance determines the string of weights in the β_i-direction. Multiplicities are a little more complicated, given by recursive formula of Freudenthal. But we will see that $\dim(V_\lambda) = \dim(V_{\sigma\lambda})$ for all $\sigma \in W$ the Weyl group associated to $\mathfrak{g}$. In particular, all the extremal weight spaces have dimension 1 (see the picture in the following example).

Example. *Let $\mathfrak{g} = \mathfrak{sl}_3$, and V be the highest weight module generated by v^+ with $hv^+ = \lambda v^+$. Then the highest weight has to be in the fundamental chamber, in a half-integral number of β_i-steps from* *each* $\mathfrak{h}_{\beta_i}$ *:*



So picking a highest weight determines all the other weights of V*:*



Formalizing this description, we have the following proposition.

Proposition 5.4. *Let* V *be a highest weight module generated by primitive* v^+ *of weight* μ*.*

(a) If V *is finite-dimensional, then* $\langle \mu, \beta^\vee \rangle \in \mathbb{Z}_{\geq 0}$ *for all* $\beta \in B$*.*

(b) If $\langle \mu, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}$*, then for each* $\alpha \in R^+$*, as a* $\mathfrak{s}_\alpha$*-module,* V *is the sum of finite-dimensional* $\mathfrak{s}_\alpha$*-modules.*

(c) The set of weights of V *is invariant under the action of* W*. In particular, there is a bijection exchanging* V_λ *and* $V_{s_\alpha(\lambda)}$*, and so* $\dim(V_\lambda) = \dim(V_{s_\alpha(\lambda)})$*.*

Proof. (a) For every $\alpha \in R^+$, let $\mathfrak{s}_\alpha$ be the corresponding submodule isomorphic to $\mathfrak{sl}_2$. Then since $x_\alpha v^+ = 0$ and $h_{\alpha^\vee} v^+ = \langle \alpha^\vee, \lambda \rangle v^+$. So if V is finite dimensional, so is $U\mathfrak{s}_\alpha v^+$, and thus

$$\langle \alpha^\vee, \lambda \rangle \in \mathbb{Z}_{\geq 0} \quad \text{ and } U\mathfrak{s}_\alpha v^+ \cong L(\langle \alpha^\vee, \lambda \rangle).$$

So this necessitates $\langle \alpha^\vee, \lambda \rangle \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in R^+$. In proving Remark 5.1, we will show as a corollary (see Corollary 6.3) that if B is a base for R, then $B^\vee = \{\beta^\vee \mid \beta \in B$ is a base for

$R^\vee = \{\alpha^\vee \mid \alpha \in R\}$ (where the definition is as you would guess, even if $R^\vee$ isn't a set of roots). So it suffices to require that $\langle \beta^\vee, \lambda\rangle \in \mathbb{Z}_{\geq 0}$ for all $\beta \in B$.

(b) Let $\hat{V}$ be the set of finite dimensional $\mathfrak{s}_\alpha$-modules in V, and let $V^\alpha = \sum_{M \in \hat{V}} M$. Since $U\mathfrak{s}_\alpha v^+$ has highest weight $\langle \mu, \alpha^\vee\rangle \in \mathbb{Z}_{\geq 0}$, $U\mathfrak{s}_\alpha v^+ \cong L(\langle \mu, \alpha^\vee\rangle)$ is finite-dimensional, and so $U\mathfrak{s}_\alpha v^+ \subseteq V^\alpha$. So V^α is non-empty. Further, for $M \in \hat{V}$, $\sum_{\gamma \in R} x_\gamma M$ is finite dimensional, and is stable under the action of $\mathfrak{s}_\alpha$ (for example,

$$x_\alpha x_\gamma M = x_\gamma x_\alpha M + [x_\alpha, x_\gamma]M \subseteq x_\gamma M + x_{\alpha+\gamma}M \subseteq \sum_{\gamma \in R} x_\gamma M),$$

so $\sum_{\gamma \in R} x_\gamma M \in \hat{V}$. Therefore, V^α is a $\mathfrak{g}$-module. But V was simple, and so $V^\alpha = V$.

(c) Let $v_\lambda \in V_\lambda \neq 0$. By the previous part, all weight vectors of V are contained in some finite sum of finite-dimensional $\mathfrak{s}_\alpha$-modules. In particular, $\langle \lambda, \alpha^\vee\rangle \in \mathbb{Z}$, and either
   (i) $\langle \lambda, \alpha^\vee\rangle = 0$, in which case $\lambda \perp \alpha$, and $s_\alpha(\lambda) = \lambda$;
   (ii) $\langle \lambda, \alpha^\vee\rangle > 0$, in which case $u_\lambda = y_\alpha^{\langle \lambda, \alpha^\vee\rangle} v_\lambda \neq 0$; or
   (iii) $\langle \lambda, \alpha^\vee\rangle < 0$, in which case $u_\lambda = x_\alpha^{\langle \lambda, \alpha^\vee\rangle} v_\lambda \neq 0$.

   In cases (ii) and (iii), the weight of u_λ (by (5.2)) is $\lambda - \langle \lambda, \alpha^\vee\rangle\alpha = s_\alpha(\lambda)$.

   The bijection follows from the fact that

$$\tau_\alpha = e^{x_\alpha} e^{-y_\alpha} e^{x_\alpha}, \quad \text{where } e^X = \sum_{n=0}^{\infty} X^n/n!,$$

   is (1) well-defined on finite-dimensional modules since y_α and x_α are nilpotent, and (2) swaps weight spaces ν and $-\nu$ in any finite-dimensional $\mathfrak{s}_\alpha$-module, since

$$\tau_\alpha \circ x_\alpha = -y_\alpha \circ \tau_\alpha, \quad \tau_\alpha \circ y_\alpha = -x_\alpha \circ \tau_\alpha, \quad \text{and} \quad \tau_\alpha \circ h_{\alpha^\vee} = -h_{\alpha^\vee} \circ \tau_\alpha.$$

   So $\dim(V_\lambda) = \dim(V_{s_\alpha(\lambda)})$ for all $\alpha \in R^+$.

□

Finally, we can describe the weights in the fundamental chamber more explicitly than just the intersection of $\frac{1}{2}\beta_i$-shifts of the $\mathfrak{h}_{\beta_i}$ hyperplanes. First, those weights are the vectors λ whose projection onto each of the simple roots β_i has a length of a half integer multiple of $||\beta_i||$. This is an integer lattice generated by λ_i, $i = 1, \ldots, r$ satisfying

$$0 = \langle \lambda_i, \beta_j\rangle \quad \text{for } i \neq j, \qquad \text{and} \qquad \tfrac{1}{2}||\alpha|| = ||\mathrm{proj}_{\beta_i}(\lambda_i)||.$$

But

$$\mathrm{proj}_{\beta_i}(\lambda_i) = \frac{\langle \beta_i, \lambda_i\rangle}{\langle \beta_i, \beta_i\rangle}\beta_i,$$

so this is the same as

$$\langle \lambda_i, \beta_j\rangle = \delta_{ij}\tfrac{1}{2}\langle \beta_j, \beta_j\rangle, \qquad \text{or} \qquad \langle \lambda_i, \beta_j^\vee\rangle = \delta_{ij} \qquad \text{where } \alpha^\vee = \frac{2}{\langle \alpha, \alpha\rangle}.$$

Note: We've seen $\alpha^\vee$ before, from the $\mathfrak{sl}_2$-triples! The set $\{\alpha^\vee \mid \alpha \in R\}$ are called the *co-roots*. The set of weights

$$\Omega = \{\omega_1, \ldots, \omega_r\} \qquad \text{satisfying} \qquad \langle \omega_i, \beta_j^\vee\rangle = \delta_{ij} \tag{5.5}$$

for $\beta_j \in B$ are called the *fundamental* weights.

The lattice of weights appearing in finite-dimensional modules, $P = \mathbb{Z}\Omega$ is called the *integral weight lattice*. If C is the fundamental chamber, then $P^+ = P \cap \bar{C}$ is called the dominant integral weight lattice, and $P \cap C$ is the strongly dominant integral weight lattice. Note that $P = WP^+$.. The weight lattice P also contains the *root lattice* $\mathbb{Z}R = \mathbb{Z}B$.

Theorem 5.5. *The simple finite-dimensional $\mathfrak{g}$-modules are highest weight modules $L(\mu)$ indexed by $\mu \in P^+ = \mathbb{Z}_{\geq 0}\Omega$.*

Proof. This almost follows from Proposition 5.4. We will need more a little more machinery concerning Weyl groups before we can prove that the set of weights of V is finite in general, but this simply amounts to the fact that there are only finitely many dominant integral weights "less than" μ. For now we will assume this. See Lemma 7.1. □

Example. *Let $\mathfrak{g} = \mathfrak{sl}_n$, with base*

$$B = \{\beta_i = \varepsilon_i - \varepsilon_{i+1} \mid i = 1, \ldots, n-1\}.$$

Then since $\langle \beta_i, \beta_i \rangle = 2$, the simple co-roots are $\beta_i^\vee = \beta_i$ and the fundamental weights are

$$\omega_i = \varepsilon_1 + \cdots + \varepsilon_i - \frac{i}{r+1}(\varepsilon_1 + \cdots + \varepsilon_{r+1}), \qquad \text{for } 1 \leq i \leq r.$$

So

$$\begin{aligned} P &= \mathbb{Z}_{\geq 0}\{\omega_1, \ldots, \omega_r\} \\ &= \{\lambda_1\varepsilon_1 + \cdots + \lambda_r\varepsilon_r - \frac{|\lambda|}{r+1}(\varepsilon_1 + \cdots + \varepsilon_{r+1})\} \end{aligned}$$

where

$$\lambda_i \in \mathbb{Z}, \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0, \qquad \text{and } |\lambda| = \lambda_1 + \cdots + \lambda_r.$$

In other words, the integral weights are in bijection with partitions $\lambda = \lambda_1, \lambda_2, \ldots$ of length $\leq n-1$, where $|\lambda|$ ranges over all non-negative integers. In short, the finite-dimensional representations of $\mathfrak{s}_n$ are indexed by partitions of length $n-1$.

Remark 5.6. What falls apart when we drop the requirement that V be finite-dimensional? (See Remark 3.2)

(1) If V is not finite-dimensional, it need not contain a primitive element. Such modules exist, and are hard to handle.
(2) If V does contain a primitive element v of weight μ, V is generated by v if and only if V is simple, and we call $V = L(\mu)$ a highest weight module of weight μ. One such module exists for every $\mu \in \mathfrak{h}^*$, but is not finite-dimensional unless $\mu \in P$.

6. More on roots and bases

Let R be the set of root associated to a semisimple finite dimensional complex Lie algebra $\mathfrak{g}$ with respect to Cartan $\mathfrak{h}$. Recall that $\mathfrak{h}^*_{\mathbb{R}} = \mathbb{R}R$ is a Euclidean space with inner product induced by $\langle , \rangle$. As before, for any weight $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$, let $\mathfrak{h}_\lambda = \{\mu \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle \mu, \lambda \rangle = 0\}$ be the hyperplane orthogonal to λ. As in (4.7), let s_α be the reflection across $\mathfrak{h}_\alpha$,

$$s_\alpha(\lambda) = \lambda - 2\text{proj}_\alpha(\lambda) = \lambda - 2\frac{\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle}\alpha.$$

Recall that if θ is the angle between two roots α and β, we have

$$\langle \alpha, \beta \rangle = ||\alpha||||\beta|| \cos(\theta) = \cos(\theta)\sqrt{\langle \alpha, \alpha \rangle \langle \beta, \beta \rangle}.$$

But recall from 4.3.1, no. 11, that $\beta(h_{\alpha^\vee}) = \langle \beta, \alpha^\vee \rangle$ is an integer for any $\alpha, \beta \in R$. So

$$\langle \beta, \alpha^\vee \rangle = 2\frac{\langle \beta, \alpha \rangle}{||\alpha||^2} = 2\frac{||\alpha||}{||\beta||}\cos(\theta)$$

and

$$\langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle = 4\cos^2(\theta)$$

are also both integers! But $0 \leq \cos^2(\theta) \leq 1$, so the only possibilities given the second equality are

$$\cos(\theta) = 0, \pm 1/2, \pm\sqrt{2}/2, \pm\frac{\sqrt{3}}{2} \pm 1.$$

Therefore, assuming $||\beta|| \geq ||\alpha||$, the only possibilities are

$\langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle$	$\langle \alpha, \beta^\vee \rangle$	$\langle \beta, \alpha^\vee \rangle$	θ
0	0	0	$\pi/2$
1	1	1	$\pi/3$
	-1	-1	$2\pi/3$
2	1	2	$\pi/4$
	-1	-2	$3\pi/4$
3	1	3	$\pi/6$
	-1	-3	$5\pi/6$

(6.1)

For the case where $\langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle = 4$, this says that $\cos^2(\theta) = 1$, so $\cos(\theta) = \pm 1$. But then $\alpha = \pm\beta$, in which case $\langle \alpha, \pm\alpha^\vee \rangle = \pm 2$. We will later return to these values for classifying all complex finite-dimensional simple Lie algebras, but first we will develop more about root systems and Weyl groups. The fact that for any two non proportional roots α and β, $\langle \alpha, \beta^\vee \rangle$ is so limited is extremely powerful, and yields unexpected results.

Lemma 6.1. *Let $\alpha, \beta \in R$ be non-proportional. If the angle θ between α and β is acute ($\langle \alpha, \beta \rangle > 0$) then $\alpha - \beta$ is also a root. Otherwise, $\alpha + \beta$ is a root.*

Proof. From the table in (7.2), one of $\langle \alpha, \beta^\vee \rangle$ or $\langle \beta, \alpha^\vee \rangle$ must be 1 if θ is acute or -1 if θ is obtuse. Suppose $\langle \alpha, \beta^\vee \rangle = \pm 1$. Then $\sigma_\beta(\alpha) = \beta \mp \alpha \in R$ with $\sigma_\beta(\alpha) = \alpha - \beta$ if θ is acute and $\sigma_\beta(\alpha) = \alpha + \beta$ if θ is obtuse. $\square$

We say a weight is *regular* if it does not sit on any of the hyperplanes determined by a root, i.e.

$$\gamma \notin \bigcup_{\alpha \in R} \mathfrak{h}_\alpha.$$

Any regular γ will choose a positive and negative set of roots for us. Namely, the hyperplane $\mathfrak{h}_\gamma$ partitions R in half, as $R = R^+ \sqcup R^-$, with

$$R^+ = R^+(\gamma) = \{\alpha \in R \mid \langle \alpha, \gamma \rangle > 0\} \quad \text{and} \quad R^- = R^-(\gamma) = \{-\alpha \mid \alpha \in R^+(\gamma)\}.$$

With respect to this decomposition, we say a root $\alpha \in R^+$ is *decomposable* if $\alpha = \beta_1 + \beta_2$ for some $\beta_1, \beta_2 \in R^+$, and that it is *indecomposable* otherwise.

Recall that a *base* B for a set of roots R is a subset of R forming a basis of $\mathfrak{h}^*$ which additionally satisfies

$$\alpha = \pm\sum_{\beta \in B} z_\beta \beta \text{ with } z_\beta \in \mathbb{Z}_{\geq 0} \quad \text{for all } \alpha \in R. \tag{6.2}$$

Proposition 6.2. *Given a regular $\gamma \in \mathfrak{h}^*_{\mathbb{R}}$, the set $B = B(\gamma)$ consisting of all indecomposable $\beta \in R^+(\gamma)$ is a base of R.*

Proof.
Each root in R^+ is a positive integral combination of elements of B.
Suppose there is some $\alpha \in R^+$ which is not a positive integral combination of elements of B. Pick one such root with $\langle \alpha, \gamma \rangle$ minimal. Trivially, $\alpha \notin B$, so $\alpha = \beta_1 + \beta_2$ for some $\beta_1, \beta_2 \in R^+$. Then

$$\langle \alpha, \gamma \rangle = \langle \beta_1, \gamma \rangle + \langle \beta_2, \gamma \rangle > \langle \beta_i, \gamma \rangle$$

for $i = 1$ and 2. So, by the minimality of $\langle \alpha, \gamma \rangle$, β_1 and β_2 are both positive integral combinations of elements of B, which is a contradiction.

B is a linearly independent set.
Recall from 4.3.1, no. 12, for any set A spanning R, $R \subseteq \mathbb{Q}A$. Suppose

$$0 = \sum_{\beta \in B} c_\beta \beta = \underbrace{\sum_{\substack{\beta \in B \\ c_\beta > 0}} p_\beta \beta}_{p} - \underbrace{\sum_{\substack{\beta \in B \\ c_\beta < 0}} n_\beta \beta}_{n},$$

where $c_\beta \in \mathbb{Q}$ and

$$p_\alpha = \begin{cases} c_\alpha & \text{if } c_\alpha > 0, \\ 0 & \text{if } c_\alpha \leq 0, \end{cases} \quad \text{and} \quad n_\alpha = \begin{cases} -c_\alpha & \text{if } c_\alpha < 0, \\ 0 & \text{if } c_\alpha \geq 0. \end{cases}$$

So that $p = n$, and have non-zero coefficients over disjoint sets of roots.

Now, let $\alpha, \beta \in B$ be distinct. By Lemma 6.1, if $\langle \alpha, \beta \rangle > 0$, then $\pm(\alpha - \beta) \in R$. But then if $\alpha - \beta \in R^+$, $\alpha = \beta + (\alpha - \beta)$ is decomposable; otherwise, $\beta = \alpha + (\beta - \alpha)$ is decomposable. Therefore $\langle \alpha, \beta \rangle \leq 0$ for distinct elements of B.

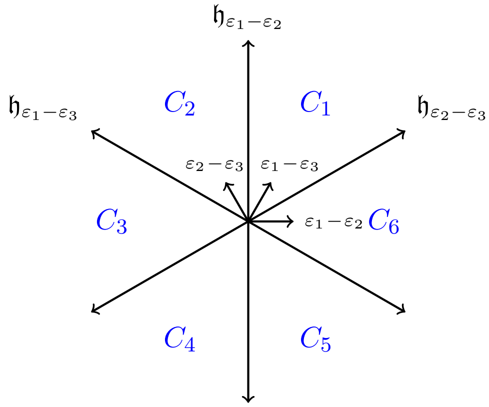
Thus

$$0 \leq \langle p, p \rangle = \langle p, n \rangle = \sum_{\beta, \beta'} p_\beta n_{\beta'} \langle \beta, \beta' \rangle \leq 0.$$

So $\langle p, p \rangle = 0$, which implies $0 = p = n$. Thus $c_\beta = 0$ for all β. □

It is additionally straightforward to show that every base B of R is of the form $B = B(\gamma)$ for any regular γ satisfying $\langle \beta, \gamma \rangle > 0$ for all $\beta \in B$ (see, for example [Hum, Theorem 2, §10.1]). So there is a base corresponding to every *Weyl chamber* (connected subset of $\mathfrak{h}_{\mathbb{R}}^* - \bigcup_{\alpha \in R}$). In particular, the regular value determining the base sits in the fundamental chamber, whose walls are exactly $\{\mathfrak{h}_\beta \mid \beta \in B\}$ (see, for example, [Bou, Ch VI, §1, no. 9, Thm 2]). In short, choosing a base, and therefore the positive roots, is the same thing as choosing a fundamental chamber.

Example. *For example, there are six different bases of the root system for $\mathfrak{sl}_3$, one for each choice of Weyl chamber:*



$$\begin{aligned}
C_1 &: B = \{\varepsilon_1-\varepsilon_2, \varepsilon_2-\varepsilon_3\}\\
C_2 &: B = \{\varepsilon_2-\varepsilon_1, \varepsilon_1-\varepsilon_3\}\\
C_3 &: B = \{\varepsilon_2-\varepsilon_3, \varepsilon_3-\varepsilon_1\}\\
C_4 &: B = \{\varepsilon_2-\varepsilon_1, \varepsilon_3-\varepsilon_2\}\\
C_5 &: B = \{\varepsilon_1-\varepsilon_2, \varepsilon_3-\varepsilon_1\}\\
C_6 &: B = \{\varepsilon_1-\varepsilon_3, \varepsilon_3-\varepsilon_2\}
\end{aligned}$$

Further, with $R^\vee = \{\alpha^\vee \mid \alpha \in R\}$, the elements of R are all scalars of elements of $R^\vee$, and so the following corollary is immediate.

Corollary 6.3. *Let $R^\vee = \{\alpha^\vee \mid \alpha \in R\}$. If $\gamma \in \mathfrak{h}^*$ is a regular weight, then $B^\vee(\gamma) = \{\beta^\vee \mid \beta \in B(\gamma)\}$ is a base of $R^\vee$ (is a basis of $\mathfrak{h}$ contained in $R^\vee$ satisfying* (6.2)*).*

A choice of fundamental chamber C, and therefore base and positive roots, also determines a *partial order* on $\mathfrak{h}^*_{\mathbb{R}}$. Namely, with $\mu, \lambda \in \mathfrak{h}^*_{\mathbb{R}}$, let

$$\lambda > \mu \quad \text{if } \lambda - \mu \text{ is the sum of simple roots.} \tag{6.3}$$

Note that this means $\alpha \in R$ is a positive root if and only if $\alpha > 0$.

Finally, a couple of loose ends.

Lemma 6.4. *Let B be a base of R.*

(1) For every $\beta, \beta' \in B$, $\langle \beta, \beta' \rangle \leq 0$ and $\beta - \beta' \notin B$.
(2) Each $\alpha \in R^+$ can be written as $\alpha = \gamma_1 + \gamma_2 + \cdots + \gamma_m$ with $\gamma_i \in B$ (not necessarily distinct) in such a way that $\gamma_1 + \cdots + \gamma_j \in R^+$ for each $1 \leq j \leq m$.

Proof. Part (1) is a direct consequence of Lemma 6.1.

For part (2), if $\alpha \in B$, then this is trivial. If not, in the proof of linear independence in Proposition 6.2, we actually showed that any set of pairwise obtuse weights sitting to one side of a hyperplane forms a linearly independent set. But since B is a basis, $\{\alpha\} \cup B$ is not linearly independent, and so $\langle \alpha, \beta \rangle > 0$ for some $\beta \in B$, and so $\alpha - \beta \in R$.

So inductively, since R is finite, α can be written as $\alpha = \gamma_1 + \gamma_2 + \cdots + \gamma_m$ with $\gamma_i \in B$ (not necessarily distinct) in such a way that $\alpha_{(j)} = \gamma_1 + \cdots + \gamma_j \in R^+$ for each $1 \leq j \leq m$. Further, since $\alpha_{(j)}$ is the positive sum of simple roots, $\beta \in R^+$.

□

6.1. **Abstract root systems, Coxeter diagrams, and Dynkin diagrams.** In light of the table in (7.2), we can now classify all possible sets of roots. Often in the literature, roots are handles completely abstractly, and then root systems associated to Lie algebras are presented as examples. Here, we have mostly avoided this abstraction until now. However some abstraction will help us with classification, so we start with some axiomatics about roots in general.

Let E be a euclidean space over $\mathbb{R}$ with inner product $\langle , \rangle$. A *symmetry* s_λ associated to $\lambda \in E$ is an automorphism of E satisfying

$$s_\lambda(\lambda) = -\lambda \qquad \text{and} \qquad E_\lambda = \{v \in E \mid s_\lambda(v) = v\} \text{ is a hyperplane in } E.$$

It is immediate that $E_\lambda^\perp = \mathbb{R}\lambda$, and s_λ has order 2 and is determined by $\mathbb{R}\lambda$. With $\lambda^\vee$ the element which is uniquely determined by

$$\langle \lambda^\vee, E_\lambda \rangle = 0 \qquad \text{and} \qquad \langle \lambda^\vee, \lambda = 2 \rangle,$$

we have $s_\lambda(\mu) = \mu - \langle \lambda^\vee, \mu \rangle \lambda$.

A subset $R \subset E$ is called a *root system* in E if

(R1) R is finite, spans E, and does not contain 0;
(R2) if $\alpha \in R$, then there is a symmetry s_α acting on E leaving R invariant; and
(R3) for each $\alpha, \beta \in R$, $s_\alpha(\beta) - \beta$ is an integer multiple of α.

Note that $s_\alpha(\alpha) = \alpha - 2\alpha = -\alpha \in R$ necessarily. Also, for any finite spanning set $R$for E, there is at most on symmetry associated to any vector λ which leaves R invariant (this follows from an analysis of eigenvalues associated to the product of any two such symmetries).

A root system is said to be *reduced* if for all $\alpha \in R$, the roots proportional to α are $\pm\alpha$. If a root system is not reduced, then it R contains a pair $\alpha, t\alpha \in R$ with $0 < t < 1$. Then (R3) forces $t = \frac{1}{2}$. So for any $\alpha \in R$, the only roots proportional to α are $\pm\alpha$ and either $\pm\frac{1}{2}\alpha$ or $\pm 2\alpha$.

Theorem 6.5. *The roots associated to finite-dimensional semisimple complex Lie algebras form reduced root systems.*

In particular, everything we've shown about roots associated to Lie algebras apply to reduced root systems. The non-reduced root systems arise for Lie algebras generated over non-algebraically closed fields, such as $\mathbb{R}$.

A subset of B a root system R is called a *base* if S is a basis for E and for all $\alpha \in R$,

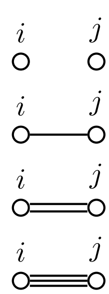
$$\alpha = \pm \sum_{\beta \in B} z_\beta \beta \qquad \text{with } z_\beta \in \mathbb{Z}_{\geq 0}.$$

The elements of B are called the simple roots. Our proof of existence of bases in the previous section holds here as well.

Let E be a euclidean space/$\mathbb{R}$ with inner product $\langle , \rangle$ (of any big dimension). Call a finite subset $A = \{\alpha_1, \ldots, \alpha_r\} \subset E$ *admissible* if

(i) A is a set of linearly independent unit vectors ($\langle \alpha_i, \alpha_i \rangle = 1$),
(ii) $\langle \alpha_i, \alpha_j \rangle \leq 0$ whenever $i \neq j$, and
(iii) $4\langle \alpha_i, \alpha_j \rangle^2 \in \{0, 1, 2, 3\}$ whenever $i \neq j$.

Associate to any admissible set A a graph $\Gamma(A)$ (called the Coxeter diagram) with vertices labeled by elements of A (or i short for α_i), with $m_{i,j} = 4\langle \alpha_i, \alpha_j \rangle^2$ edges connecting i to j:



if $4\langle \alpha_i, \alpha_j \rangle^2 = 0$,

if $4\langle \alpha_i, \alpha_j \rangle^2 = 1$,

if $4\langle \alpha_i, \alpha_j \rangle^2 = 2$,

if $4\langle \alpha_i, \alpha_j \rangle^2 = 3$.

Note that by normalizing the elements of any base B for a set of roots R, you get an admissible set A. The *Coxeter diagram* associated to a root system R is the graph associated to the normalization of a base B. In particular, the diagram does not depend on the chosen base (we will see that the Weyl group acts transitively on Weyl chambers, and so all Weyl chambers have the same set of angles between pairs of chamber walls).

Theorem 6.6 ([Ser, §11,12])**.** *Every root system is a sum of irreducible root systems, and a root system is irreducible if and only if the associated Coxeter diagram is connected.*

This follows because R_1 decomposes into R_1 and R_2 if and only if R_1 and R_2 are orthogonal to each other, in which case there are no edges connecting to the vertices for R_1 to the vertices for R_2.

A *Dynkin diagram* associated to a base B for a root system is a decorated Coxeter graph for the associated normalized admissible set. If α_i is adjacent to α_j, and the root β_i associated to α_i is longer than the root β_j associated to α_j, decorate the $m_{i,j}$ edges connecting α_i to α_j with an arrow pointing to α_i (the normalization of the longer root).

Suppose that E is the direct sum of (non-trivial) subspaces E_i, $i = 1, \dots, \ell$, such that $R \subset \bigcup_{i=1}^{\ell} E_i$. Then $R_i = E_i \cap R$ is a root system for E_i. We say R is the sum of subsystems R_i. If the only decompositions trivial ($\ell = 1$), we say R is *irreducible*. For example, the root systems associated to simple Lie algebras are irreducible, while semisimple but not simple Lie algebras have reducible root systems.

The *Cartan matrix* associated with R is the matrix $(\langle \alpha, \beta^\vee \rangle)_{\alpha,\beta \in B}$. In particular, since $\langle \alpha, \alpha^\vee \rangle = 2$, the diagonal entries are all 2, and since $\langle \alpha, \beta \rangle \leq 0$ for $\alpha \neq \beta$, the off-diagonal entries are all $0, -1, -2$, or -3.

Proposition 6.7 ([Ser, §11, Prop 8 and §15, Prop 13])**.** *A reduced root system is determined (up to isomorphism) by its Cartan matrix and vice versa. A root system is also determined (up to isomorphism) by its dynkin diagram.*

Then due to the triangular decomposition of any finite-dimensional semisimple Lie algebra, the Lie algebra is determined by its root system. So to classify all simple finite-dimensional Lie algebras, one must simply classify all connected Dynkin diagrams (of finite type). On the homework, you're asked to (1) narrow down the possible connected Coxeter diagrams, (2) show existence of admissible sets for the leftover graphs, and (3) classify the connected Dynkin diagrams (of finite type).

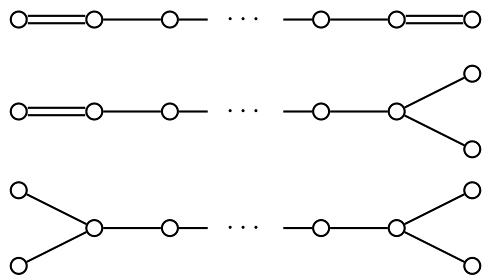
Exercise 6: Some things about classification.

(1) Let $A = \{\alpha_1, \dots, \alpha_r\}$ be an admissible set yielding a connected graph $\Gamma(A)$.
   (a) Show that the number of pairs of vertices connected by at least one edge strictly less than r.
   [What is the condition on vertices being adjacent? Consider $\langle \alpha, \alpha \rangle$ where $\alpha = \sum_A \alpha_i$.]
   (b) Show that $\Gamma(A)$ contains no cycles. [Note that any subset of an admissible set is admissible.]
   (c) Show that the degree (counting multiple edges) of any vertex in $\Gamma(A)$ is no more than three.
   [Take a vertex $\alpha \in A$, and let S be the set containing α together with its neighborhood (the vertices adjacent to it). Note that in the span of S is a unit vector β which is orthogonal to $S - \{\alpha\}$, so that $\alpha = \sum_{\gamma \in S - \{\alpha\} + \{\beta\}} \langle \alpha, \gamma \rangle \gamma$ and $\langle \alpha, \beta \rangle \neq 0$ (why??).]
   (d) Show that if $S \subseteq A$ has graph

$$\Gamma(S) = \circ\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!\circ\!\!-\!\!- \ \cdots \ -\!\!-\!\!\circ\!\!-\!\!-\!\!\circ,$$

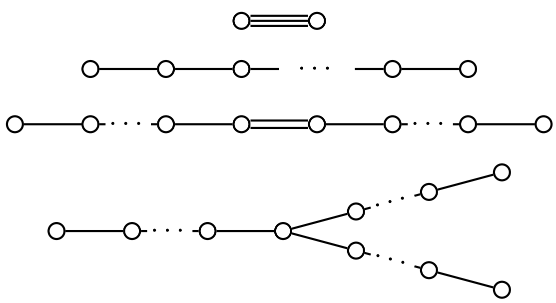
then $A' = A - S + \{\sum_S \alpha\}$ is admissible (with graph $\Gamma(A')$ obtained by collapsing the subgraph $\Gamma(S)$ to a single vertex).

(e) Show that $\Gamma(A)$ cannot contain any of the following graphs as subgraphs:

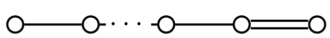
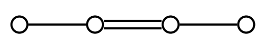


[Use the previous part]

(f) Show that the only remaining possible graphs associated to admissible sets are of one of the following four forms:

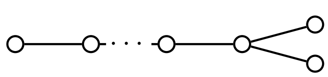
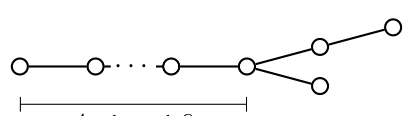


(g) Show the only possible graphs of the third type are

 and 

[Suppose the vectors corresponding to the vertices to the left of the double bond are $\lambda_1, \ldots, \lambda_\ell$ (from left to right) and the vertices to the rights of the double bond are $\mu_1, \ldots, \mu_m$ (from right to left). Let $\lambda = \sum_i i\lambda_i$ and $\mu = \sum_i i\mu_i$. Show that $\langle \lambda, \lambda \rangle = \ell(\ell+1)/2$, $\langle \mu, \mu \rangle = m(m+1)/2$, and $\langle \lambda, \mu \rangle^2 = \ell^2 m^2/2$, and use the Cauchy-Schwarz inequality for inner products.]

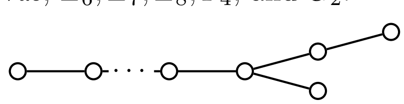
(h) Bonus: Show the only graphs of the fourth kind are

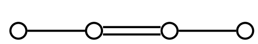
 and 

$4 \leq * \leq 6$

[This is like the previous part, only more so]

(2) Show that there's an admissible set associated to every remaining graph by displaying existence. Namely, associate most of the remaining possible graphs to a classical root systems (showing existence), and take for granted that the remaining five are associated to the *exceptional simple Lie algebras*, E_6, E_7, E_8, F_4, and G_2:

E_6, E_7, E_8 : 

F_4 : 

G_2 : 

(3) Classify all (finite type) connected Dynkin diagrams.

7. Weyl groups

Now that we're armed with all this great structure on R and understand a little more about Weyl chambers and roots, let's explore some properties of Weyl groups.

Fix a fundamental chamber C, and therefore a base B and positive set of roots R^+. With $B = \{\beta_1, \dots, \beta_r\}$, let $s_i = s_{\beta_i}$. Let W be the group generated by $\{s_\alpha \mid \alpha \in R\}$.

Lemma 7.1.

(1) The Weyl group W is finite.

(2) The form $\langle , \rangle$ on $\mathfrak{h}_{\mathbb{R}}^*$ is W-invariant, i.e.

$$\langle w(\alpha), \beta \rangle = \langle \alpha, w^{-1}(\beta) \rangle, \qquad \text{for all } \alpha, \beta \in R, w \in W.$$

(3) For all $\alpha \in R$, $w \in W$,

$$w s_\alpha w^{-1} = s_{w(\alpha)}.$$

Also, $w(\alpha^\vee) = w(\alpha)^\vee$.

(4) The reflection associated to a simple root β setwise fixes $R^+ - \{\beta\}$ and $R^- - \{-\beta\}$.

(5) If $w = s_{i_1} s_{i_2} \cdots s_{i_{\ell-1}}$ sends β_{i_ℓ} to a negative root, then $w s_{i_\ell} = s_{i_1} \cdots s_{i_{m-1}} s_{i_{m+1}} \cdots s_{i_{\ell-1}}$ for some $1 \le m < \ell$.

(6) If $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ with ℓ minimal, then $w(\beta_{i_\ell}) < 0$.

Proof. (1) Since W a reflection group in $\mathrm{GL}(\mathfrak{h}_{\mathbb{R}}^*)$ which is defined by its action on R, it can be identified with a subgroup of S_R, the group of permutations on the set R. But R is finite, so W is finite.

(2) Reflections are rigid motions, and so preserve lengths of and angles between vectors. So since

$$\langle \alpha, \beta \rangle = ||\alpha|| ||\beta|| \cos(\alpha \angle \beta),$$

we have $\langle w\alpha, w\beta \rangle = \langle \alpha, \beta \rangle$. Further, since $w(c\alpha) = cw(\alpha)$, we have

$$w(\alpha^\vee) = w\left(\frac{2}{\langle \alpha, \alpha \rangle} \alpha \right) = \frac{2}{\langle \alpha, \alpha \rangle} w(\alpha) = \frac{2}{\langle w(\alpha), w(\alpha) \rangle} w(\alpha) = w(\alpha)^\vee.$$

(3) Notice for any $\gamma \in R$, since $w(\gamma) \in R$, so is

$$w s_\alpha w^{-1}(w(\gamma)) = w s_\alpha(\gamma) = w(\gamma - \langle \gamma, \alpha^\vee \rangle \alpha) = w(\gamma) - \langle \gamma, \alpha^\vee \rangle w(\alpha).$$

Since reflections are 1-1, as γ runs over R, so does $w(\gamma)$. So $w s_\alpha w^{-1}$ is the reflection which acts by

$$w s_\alpha w^{-1} \gamma = \gamma - \langle w^{-1}(\gamma), \alpha^\vee \rangle w(\alpha) = \gamma - \langle \gamma, w(\alpha)^\vee \rangle w(\alpha)$$

on all γ, it must be the reflection $s_{w(\alpha)}$. (See [Hum, Lem. 1] for a detailed proof of this last conclusion).

(4) Let $\alpha \in R^+$ and write $\alpha = \sum_{\gamma \in B} z_\gamma \gamma$ with $z_\gamma \in \mathbb{Z}_{\ge 0}$. So

$$\begin{aligned} \sigma_\beta(\alpha) &= \alpha - \langle \alpha, \beta^\vee \rangle \beta \\ &= (z_\beta - \langle \alpha, \beta^\vee \rangle)\beta + \sum_{\gamma \in B - \{\beta\}} z_\gamma \gamma. \end{aligned}$$

Therefore, either $\alpha = \beta$, in which case $\sigma_\beta(\beta) = -\beta \in R^-$, or $\alpha \neq \beta$, so that $z_\gamma \neq 0$ for some $\gamma \neq \beta$. In the later case, since the coefficient of some $\gamma \in B$ in $\sigma_\beta(\alpha)$ is positive, all of them are. So $\sigma_\beta(\alpha) \in R^+$.

(5) Let $w_j = s_{i_j} \cdots s_{i_{\ell-1}}$ so that $w_\ell = 1$ and $w_1 = w$. But since $w_1(\beta_{i_\ell}) < 0$ and $w_\ell(\beta_{i_\ell}) > 0$, there is some index where $w_{m+1}(\beta_{i_\ell})$ is positive and $w_m(\beta_\ell) = s_{i_m} w_{m+1}(\beta_{i_\ell})$ is negative. But by part (4), the only positive root that gets sent to a negative root by the simple reflection s_{i_m} is β_{i_m}. So $w_{m+1}(\beta_{i_\ell}) = \beta_{i_m}$. Therefore, by part (3),

$$s_{i_m} = w_{i_{m+1}} s_{i_\ell} w_{i_{m+1}}^{-1} = s_{i_{m+1}} \cdots s_{i_{\ell-1}} s_{i_\ell} s_{i_{\ell-1}} \cdots s_{i_{m+1}},$$

so that

$$w s_{i_\ell} = s_{i_1} \cdots s_{i_{m-1}} s_{i_m} s_{i_{m+1}} \cdots s_{i_\ell} = s_{i_1} \cdots s_{i_{m-1}} s_{i_{m+1}} \cdots s_{i_{\ell-1}}.$$

(6) Either $w(\beta_{i_\ell}) < 0$, or $ws_{i_\ell}(\beta_{i_\ell}) < 0$. If $ws_{i_\ell}(\beta_{i_\ell}) < 0$, then part (5) applied to $ws_{i_\ell} = s_{i_j} \cdots s_{i_{\ell-1}}$ says that w can be written as a shorter word, which is a contradiction.

□

A direct consequence of Lemma 7.1(4) is that with

$$\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha, \tag{7.1}$$

we have $s_\beta(\rho) = \rho - \beta$. Notice that $s_\beta(\rho) = \rho - \beta$ implies that $\langle \rho, \beta^\vee \rangle = 1$ for all $\beta \in B$. So $\rho = \sum_{i=1}^r \omega_i$, and ρ is a (strongly) dominant integral weight.

Example. *In the case where $\mathfrak{g}$ is type A_r, let $B = \{\beta_i = \varepsilon_i - \varepsilon_{i+1} \mid i = 1, \dots, r\}$, so that $\omega_i = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_i - \frac{i}{r+1}(\varepsilon_1 + \cdots + \varepsilon_{r+1})$. So*

$$\begin{aligned}
\rho &= \tfrac{1}{2} \sum_{i<j} \varepsilon_i - \varepsilon_j \\
&= \tfrac{1}{2} \left((r-0)\varepsilon_1 + (r-1-1)\varepsilon_2 + (r-2-2)\varepsilon_3 + \dots (0-r)\varepsilon_{r+1} \right) \\
&= \tfrac{1}{2} \sum_{i=1}^{r+1} (r+2-2i)\varepsilon_i \\
&= \sum_{i=1}^{r} (\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_i) - \frac{1}{r} \left(\sum_{i=1}^{r} i \right) (\varepsilon_1 + \cdots + \varepsilon_{r+1}) \\
&= \sum_{i=1}^{r} \omega_i.
\end{aligned}$$

Now for the big theorem on Weyl groups!

Theorem 7.2.

(1) W acts transitively on Weyl chambers.
(2) Fix a base B. For all $\alpha \in R$ there is some $w \in W$ with $w(\alpha) \in B$.
(3) For any base B, W is generated by simple reflections (reflections associated to simple roots).
(4) W acts simply transitively on bases B of R.

Proof. Let $G \in W$ be the group generated by the simple reflections. Since W is finite, so is G.

(1) Fix a fundamental chamber C, and therefore a base B. Let C' be any Weyl chamber and let $\gamma \in C'$ be a regular element. We will show that there is some $w \in G$ sending C' to C.

With ρ as in (7.1), pick $w \in G$ with $\langle w(\gamma), \rho\rangle$ maximal. Then for all $\beta \in B$,

$$\begin{aligned}\langle w(\gamma), \rho\rangle \geq \langle s_\beta w(\gamma), \rho\rangle &= \langle w(\gamma), s_\beta(\rho)\rangle \\ &= \langle w(\gamma), \rho\rangle - \langle w(\gamma), \beta\rangle.\end{aligned}$$

So $\langle w(\gamma), \beta\rangle \geq 0$ for all $\beta \in B$. But γ was regular, and so $w(\gamma)$ is as well. Therefore $\langle w(\gamma), \beta\rangle > 0$ for all $\beta \in B$, and so $w(\gamma) \in C$. So w sends C' to C.

(2) Since G acts transitively on chambers, it acts transitively on bases as well. So it suffices to show every root α lies in some base. To show that, we can find a regular γ so that $\alpha \in B(\gamma)$. Any regular γ close enough to the hyperplane $\mathfrak{h}_\alpha$ (depending on its distance from the origin) and on the same side as α so that $|\langle \alpha', \gamma\rangle| \geq \langle \alpha, \gamma\rangle > 0$ for all $\alpha' \neq \pm\alpha$ will suffice.

(3) For a fixed base B, let $w \in G$ send some fixed $\alpha \in R$ into B, i.e. $w(\alpha) = \beta \in B$. Then by Lemma 7.1(3), $s\alpha = ws_\beta w^-1 \in G$. So $W \subseteq G \subseteq W$, and thus $G = W$.

(4) Again, since since $G = W$ acts transitively on chambers, it acts transitively on bases as well. So we only need show that

$$w(B) = B \quad \text{if any only if} \quad w = 1.$$

But by writing w as a minimal product of simple reflections (by part (3)), this is a direct result of Lemma 7.1(6).

□

So the Weyl group is generated by reflections coming from the base $B = \{\beta_1, \ldots, \beta_r\}$. The relations are given by the angles between the corresponding hyperplanes. Let $n_{i,j} = \langle \beta_i, \beta_j^\vee\rangle\langle \beta_i^\vee, \beta_j\rangle$, $\theta_{i,j} = \beta_i \angle \beta_j$, and $s_i = s_{\beta_i}$. So returning to table (7.2), we have that the relation

$$\underbrace{s_i s_j \cdots}_{m_{i,j} \text{ terms}} = \underbrace{s_j s_i \cdots}_{m_{i,j} \text{ terms}},$$

where $m_{i,j}, n_{i,j}$, and $\theta_{i,j}$ are connected as

$$\theta_{i_j} = \arccos(\pm\sqrt{n_{i,j}/4}) = \pi/m_{i,j} :$$

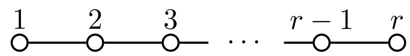
Coxeter subgraph	$n_{i,j}$	$\theta_{i,j}$	$m_{i,j}$
$\overset{i}{\circ}\quad\overset{j}{\circ}$	0	$\pi/2$	2
$\overset{i}{\circ}\!-\!\!\!-\!\!\!-\!\overset{j}{\circ}$	1	$\pi/3$	3
$\overset{i}{\circ}\!=\!\!=\!\overset{j}{\circ}$	2	$\pi/4$	4
$\overset{i}{\circ}\!\equiv\!\!\equiv\!\overset{j}{\circ}$	3	$\pi/6$	6

(7.2)

Example. *Let $\mathfrak{g} = A_r$. One base for $R = \{\pm\alpha_{i,j} = \pm(\varepsilon_i - \varepsilon_j) \mid 1 \leq i < j \leq r\}$ is $B = \{\beta_i = \varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq r\} = B^\vee$. Since*

$$\langle \beta_i, \beta_{i\pm1}^\vee\rangle = -1 \qquad \textit{and} \qquad \langle \beta_i, \beta_j^\vee\rangle = 0 \textit{ for } j \neq i \pm 1,$$

the Coxeter diagram looks like



So the Weyl group is

$$W = \langle s_1, \dots, s_r \mid s_i^2 = 1, s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, s_i s_j = s_j s_i \text{ for } i \neq j \pm 1 \rangle \cong S_{r+1},$$

the symmetric group on $r+1$ letters.

Example. *The lie algebras B_r and C_r have different root systems (because they're not isomorphic), so they have different Dynkin diagrams. But they have the same Coxeter diagram, and therefore the same Weyl group. This is because the difference between their root systems comes in the relative lengths of the roots, not in the angles between them. Nice bases for B_r and C_r, respectively, are*

$$B_{B_r} = \{\beta_0 = \varepsilon_1, \beta_i = \varepsilon_{i+1} - \varepsilon_i \mid 1 \leq i \leq r-1\}$$

and

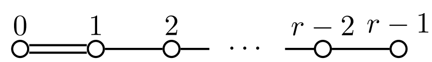
$$B_{C_r} = \{\beta_0 = 2\varepsilon_1, \beta_i = \varepsilon_{i+1} - \varepsilon_i \mid 1 \leq i \leq r-1\},$$

with $B_{B_r}^\vee = B_{C_r}$ and $B_{C_r}^\vee = B_{B_r}$. We have

$$\langle \beta_i, \beta_{i+1}^\vee \rangle = -1 \text{ for } i \geq 1, \qquad \langle \beta_i, \beta_j^\vee \rangle = 0 \text{ for } j \neq i \pm 1,$$

$$\langle 2\varepsilon_1, \varepsilon_1 \rangle = 2.$$

So since $\varepsilon_1^\vee = 2\varepsilon_1$ and $(2\varepsilon_1)^\vee = \varepsilon_1$, both B_r and C_r have Coxeter diagrams



So the Weyl group of type BC_r is

$$W = \left\langle s_1, \dots, s_r \;\middle|\; \begin{array}{c} s_i^2 = 1, s_i s_j = s_j s_i \text{ for } i \neq j \pm 1 \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ for } i \geq 1 \\ s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0 \end{array} \right\rangle \cong Z_2 \ltimes S_r,$$

the group of signed permutations on r letters (the subgroup generated by $s_1, \dots, s_{r-1}$ is the group of permutations; then let s_0 act by flipping the sign of the first element in the permutation).

Given that W is generated by simple reflections with various relations, there might be several ways to write a given element of W. So the length of any expression $w = s_{i_1} s_{i_2} \dots s_{i_m}$ might change, but there is certainly at least one shortest expression. Define the *length* $\ell(w)$ of an element $w \in W$ to be the length of the shortest expression of w in terms of simple reflections. For example, in type A_r as in Example 7,

$$s_1 s_2 s_1 = s_2 s_1 s_2, \qquad \text{so} \qquad \ell(s_1 s_2 s_1 s_1) = \ell(s_2 s_1 s_2^2) = \ell(s_2 s_1) = 2,$$

since there is no shorter expression for $s_2 s_1$. While it may take some work to calculate the length of any given $w \in W$, any two expressions for w will have the same parity in length. Namely, the possible relations in W, given by

$$s_i^2 = 1, \qquad \text{taking a length 2 expression to a length 0 expression, and}$$

$$\underbrace{s_i s_j \cdots}_{m_{i,j} \text{ terms}} = \underbrace{s_j s_i \cdots}_{m_{i,j} \text{ terms}}, \qquad \text{taking a length } m_{i,j} \text{ expression to a length } m_{i,j} \text{ expression,}$$

both preserve parity. More broadly, one can still see the same parity by generating w with any root reflections because all root reflections are conjugate to simple reflections (as we saw in Theorem 7.2(3)). So the map

$$\det : W \to \{\pm 1\} \quad \text{given by} \quad \det(w) = \begin{cases} 1 & \text{if } w \text{ is the product of an even number of reflections,} \\ -1 & \text{if } w \text{ is the product of an odd number of reflections,} \end{cases} \tag{7.3}$$

is well defined. In fact, this is what we call the *alternating representation* or the *sign representation* of W. In the case where $W = S_{r+1}$, this is the representation indexed by the single column partition (1^{r+1}) of $r+1$. In the literature, this map most often goes by the notation $\det(w)$ or $\varepsilon(w)$.

For each Weyl group, there is a unique longest word w_0 (i.e. $\ell(w_0) > w$ for all $w \in W - \{w_0\}$). This is the element which sends the fundamental chamber C to its opposite, so that $w_0 C$ is the unique Weyl chamber on the negative side of all hyperplanes (corresponding to the base $-B$). This is also the map which takes ρ to $-\rho$.

8. Back to representation theory

We know from section 5 that the finite dimensional simple $\mathfrak{g}$-modules $L(\lambda)$ are highest weight modules with highest weight in P^+, the set of weights λ satisfying $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in R^+$. We know they're generated by a primitive v_λ^+, and spanned by $\{y_1^{\ell_1} \cdots y_m^{\ell_m} v_\lambda^+ \mid R^+ = \{\alpha_1, \ldots, \alpha_m\}, y_i \in \mathfrak{g}_{-\alpha_i}, \ell_i \in \mathbb{Z}_{\geq 0}\}$ But which of these are lineally independent? In other words, what are the dimensions go the weight spaces in $L(\lambda)$?

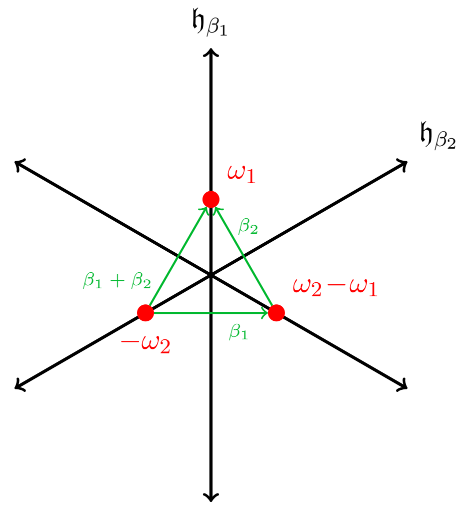
We have a few clues. First, we know which weight spaces are non-trivial. For any finite-dimensional $\mathfrak{g}$-module, let $P_V = \{\lambda \in P \mid V_\lambda \neq 0\}$ be the set of weights of V; when $V = L(\lambda)$, write P_λ for short. We learned in Proposition 5.4 that P_λ consists of integral weights μ in the convex hull of $W\lambda$ which are *congruent* to λ modulo R^+, i.e.

$$\mu = \lambda - \sum_{\alpha \in R^+} \ell_\alpha \alpha.$$

We also learned that $\dim(L(\lambda)_\mu) = \dim(L(\lambda)_{w\mu})$ for all $w \in W$.

So one circumstance where we know the exact dimensions is when $P_\lambda = W_\lambda$, i.e. where all weights in $L(\lambda)$ are in the W-orbit of λ. We call such weights λ *minuscule*, referring to the fact that they are the weights of P^+ closest to the origin. Some people include 0 in the set of minuscule weights, some do not.

Example. *Let $\mathfrak{g} = A_2$. The weight ω_1 has W-orbit $\{\omega_1, -\omega_2, \omega_2 - \omega_1\}$, whose convex hull contains no other R-shifts:*



So ω_1 is miniscule. In fact, the (non-trivial) minuscule weights of A_2 are exactly ω_1 and ω_2.

For type A_r, all of the fundamental weights. In general, the minuscule weights are a subset of the fundamental weights.

Have caution, though: the minuscule weights are not always the same as the fundamental weights! For type C_r, one base is $B = \{\beta_i \mid i = 1, \dots, r\}$, with fundamental weights given by

$$\omega_i = \varepsilon_1 + \cdots + \varepsilon_i \quad \text{for } i = 1, \dots, r.$$

Notice that $\varepsilon_1 + \varepsilon_2$ is both a fundamental weight and a root, so that 0 is a weight of $L(\omega_2)$ and so ω_2 is not miniscule.

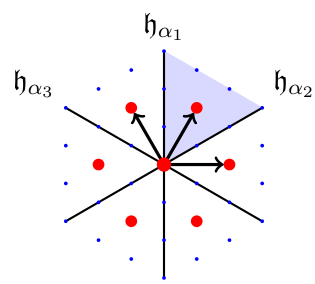
To study the dimensions of the weight spaces in general, we need more information.

One way to answer the question would be to ask for a weight basis. We have a weight spanning set–one first guess is that it's more or less a basis. In other words, we should ask if

$$\{y_1^{\ell_1} \cdots y_m^{\ell_m} v_\lambda^+ \mid y_i^{\ell_i - j} y_{i+1}^{\ell_{i+1}} \cdots y_m^{\ell_m} v_\lambda^+ \neq 0 \ \forall i, j\}$$

(with appropriate conditions on j) is a basis. Unfortunately, it is not. Let's look at an example to see why it can't be.

Example. *Let $\mathfrak{g} = A_2$ have base $B = \{\beta_1, \beta_2 \mid \beta_i = \varepsilon_i - \varepsilon_{i+1}\}$, so that $R^+ = \{\alpha_1 = \beta_1, \alpha_2 = \beta_2, \alpha_3 = \beta_1 + \beta_2\}$. Let $\lambda = \alpha_3$. The P_λ are the red points in*



so that

$$P_{\alpha_3} = W\alpha_3 \sqcup \{0\}, \quad \text{where} \quad W\alpha_3 = R.$$

Then the set $\{y_1^{\ell_1}\cdots y_m^{\ell_m}v_\lambda^+ \mid \lambda - \sum_{i=1}^m \ell_m\alpha_m \in P_\lambda\}$ *has*

$$\begin{array}{rll}
v_\lambda^+ & \textit{with weight} & \boxed{\alpha_3} \\
y_1v_\lambda^+ & \textit{with weight} & \alpha_3-\alpha_1=\boxed{\alpha_2} \\
y_1y_2v_\lambda^+ & \textit{with weight} & \alpha_3-\alpha_1-\alpha_2=\boxed{0} \\
y_1^2y_2v_\lambda^+ & \textit{with weight} & \alpha_3-2\alpha_1-\alpha_2=\boxed{-\alpha_1} \\
y_1y_2y_3v_\lambda^+ & \textit{with weight} & \alpha_3-\alpha_1-\alpha_2-\alpha_3=\boxed{-\alpha_3} \\
y_1y_3v_\lambda^+ & \textit{with weight} & \alpha_3-\alpha_1-\alpha_3=\boxed{-\alpha_1} \\
y_2v_\lambda^+ & \textit{with weight} & \alpha_3-\alpha_2=\boxed{\alpha_1} \\
y_2y_3v_\lambda^+ & \textit{with weight} & \alpha_3-\alpha_2-\alpha_3=\boxed{-\alpha_2} \\
y_3v_\lambda^+ & \textit{with weight} & \alpha_3-\alpha_3=\boxed{0} \\
y_3^2v_\lambda^+ & \textit{with weight} & \alpha_3-2\alpha=\boxed{-\alpha_3}
\end{array}$$

But we know the dimension of the weight spaces corresponding to $-\alpha_1$ *and* $-\alpha_3$ *must be the same as that of* α_3*, which is 1, so this can't be a linearly independent set. We to know that the multiplicity of 0 can be at most 2, though, since this is a spanning set.*

We'll explore three (maybe more) ways of getting at the actual dimensions, mostly without proof for the sake of time. I will, however, point to references and walk us though the main ideas to facilitate reading those references in the future.

8.1. **The Universal Casimir element and Freudenthal's multiplicity formula.** Freudenthal's multiplicity formula is a recursive formula for calculating the dimensions of the weight spaces in a highest weight module $L(\lambda)$. The proof relies on calculating the trace of a particular central element of $U\mathfrak{g}$ on each weight space. You can find such proofs, for example, in [Hum, §22.3], [FH, §22.1], or [Bou, §VIIII.9.3] (in disguise). The particular central element they rely on is the universal Casimir element you saw on the homework.

Namely, if $\{b_i\}$ is a basis of $\mathfrak{g}$, then there is a unique dual basis $\{b_i^*\}$ of $\mathfrak{g}$ determined by $\langle b_i, b_i^*\rangle = \delta_{ij}$.The *Casimir element* is

$$\kappa = \sum_{b_i} b_ib_i^* \in U\mathfrak{g}$$

where the sum is over the basis $\{b_i\}$ and the dual basis $\{b_i^*\}$.

Theorem 8.1. *Let* κ *be the Casimir element of* $\mathfrak{g}$*.*

(1) κ *does not depend on the choice of basis.*
(2) $\kappa \in \mathcal{Z}(U\mathfrak{g})$*, the center of* $U(\mathfrak{g})$*.*

Proof.

(1) Note first that $\{b_1^*, \ldots, b_\ell^*\}$ is also a basis of $\mathfrak{g}$. Let $\{d_1, \ldots, d_\ell\}$ be a third basis of $\mathfrak{g}$. Then $b_i = \sum_j \langle b_i, d_j^* \rangle d_j$ implies

$$\begin{aligned}\kappa = \sum_{i=1}^{\ell} b_i b_i^* &= \sum_{i,j=1}^{\ell} \langle b_i, d_j^* \rangle d_j b_i^* \\ &= \sum_{j=1}^{\ell} d_j \left(\sum_i \langle b_i, d_j^* \rangle b_i^* \right) = \sum_{j=1}^{\ell} d_j d_j^*.\end{aligned}$$

(2) Let $x \in \mathfrak{g}$. Then

$$\begin{aligned}x\kappa = \sum_{i=1}^{\ell} x b_i b_i^* &= \sum_{i=1}^{\ell} ([x, b_i] + b_i x) b_i^* \\ &= \sum_{i,j=1}^{\ell} \langle [x, b_i], b_j^* \rangle b_j b_i^* + \sum_{i=1}^{\ell} b_i x b_i^* \\ &= -\sum_{i,j=1}^{\ell} \langle b_i, [x, b_j^*] \rangle b_j b_i^* + \sum_{i=1}^{\ell} b_i x b_i^* \\ &= -\sum_{j=1}^{\ell} b_j [x, b_j^*] + \sum_{i=1}^{\ell} b_i x b_i^* \\ &= \sum_{i=1}^{\ell} b_i (-x b_i + b_i x + x b_i) = \kappa x.\end{aligned}$$

□

Since κ is central, Schur's lemma ensures that κ acts by a constant on any simple $\mathfrak{g}$-module (since $M \to M$ by $m \to \kappa m$ is a $\mathfrak{g}$-module isomorphism). So the trace of the action of κ on a weight space $L(\lambda)_\mu$ will be equal to $\kappa_\lambda m_\mu$ where κ_λ is the constancy by which κ acts on $L(\lambda)$ and $m_\mu = \dim(L(\lambda)_\mu)$.

On the homework, you will be asked to show that

$$\kappa L(\lambda) = \kappa_\lambda L(\lambda) \qquad \text{where} \kappa_\lambda = \langle \lambda + \rho, \lambda + \rho \rangle - \langle \rho, \rho \rangle = \langle \lambda, \lambda + 2\rho \rangle \tag{8.1}$$

Then the following theorem essentially amounts to calculating the trace of the action of κ on $L(\lambda)_\mu$ another way, which ends up being a recursive process.

Theorem 8.2 (Freudenthal's multiplicity formula)**.** *Let m_μ be the dimension of $L(\lambda)_\mu$ in $L(\lambda)$, with $\lambda \in P^+$. Then m_μ is determined recursively by*

$$m_\mu = \frac{2}{\langle \lambda, \lambda + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle} \sum_{\alpha \in R^+} \sum_{i=1}^{\infty} \langle \mu + i\alpha, \alpha \rangle m_{\mu + i\alpha}.$$

Example. *Let's return to example 8. First let's do a sanity check, and verify that Theorem 8.2 says, for example, that $m_{\alpha_2} = 1$ as it should, i.e.*

$$1 = m_{\alpha_2} = \frac{2}{\langle \alpha_3, \alpha_3 + 2\rho \rangle - \langle \alpha_2, \alpha_2 + 2\rho \rangle} \sum_{j=1}^{3} \sum_{i=1}^{\infty} \langle \mu + i\alpha_j, \alpha_j \rangle m_{\mu + i\alpha_j}.$$

Here $\rho = \frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3) = \alpha_3$ *(since* $\alpha_3 = \alpha_1 + \alpha_2$*), so*

$$\frac{2}{\langle \alpha_3, \alpha_3 + 2\rho\rangle - \langle \alpha_2, \alpha_2 + 2\rho\rangle} = \frac{2}{3\langle \alpha_3, \alpha_3\rangle - \langle \alpha_2, \alpha_2 + 2\alpha_3\rangle} = \frac{2}{3 * 2 - (2+2)} = \frac{2}{2} = 1.$$

Next, the only positive root shift of α_2 *which is in* P_{α_3} *is* $\alpha_2 + \alpha_1 = \alpha_3$*, so*

$$m_{\alpha_2} = 1 * (\langle \alpha_2 + \alpha_1, \alpha_1\rangle m_{\alpha_3}) = 1 * (1) = 1.\checkmark$$

Now, the only multiplicity that was actually in question is that of $\mu = 0$*. Theorem 8.2 says*

$$m_0 = \frac{2}{\langle \alpha_3, \alpha_3 + 2\rho\rangle - \langle 0, 0 + 2\rho\rangle} \sum_{j=1}^{3} \sum_{i=1}^{\infty} \langle \mu + i\alpha_j, \alpha_j\rangle m_{\mu + i\alpha_j}.$$

First,

$$\frac{2}{\langle \alpha_3, \alpha_3 + 2\rho\rangle - \langle 0, 0 + 2\rho\rangle} = \frac{2}{3\langle \alpha_3, \alpha_3\rangle} = \frac{2}{3 * 2} = \frac{1}{3}.$$

Next, the only positive root shifts of 0 which are in P_{α_3} *are where* $i = 1$ *in the sum above. So*

$$\begin{aligned} m_0 &= \frac{1}{3}(\langle 0 + \alpha_1, \alpha_1\rangle m_{0+\alpha_1} + \langle 0 + \alpha_2, \alpha_2\rangle m_{0+\alpha_2} + \langle 0 + \alpha_3, \alpha_3\rangle m_{0+\alpha_3}) \\ &= \frac{1}{3}(2 + 2 + 2) = 2. \end{aligned}$$

So the multiplicity of the weight 0 in $L(\alpha_3)$ *is 2 after all!*

8.2. **Weyl character formula.** Sometimes in representation theory, instead of studying the representations head on, we study their characters instead. Back when we were reviewing the properties of the symmetric group on the first day, I mentioned that characters of groups were class functions, and that irreducible characters were in one-to-one correspondence with irreducible representations. They also satisfy really nice additive and multiplicative properties. So really, characters are functions that are built to contain all the relevant representation theoretic data. Here, we embark on studying the characters of finite-dimensional $\mathfrak{g}$-modules.

As before, let $P = \mathbb{Z}\{\omega_1, \ldots, \omega_r\}$ (with ω_i as in (5.5)) be the integral weights of $\mathfrak{g}$, i.e. the indexing set for the weights appearing in finite-dimensional representations of $\mathfrak{g}$. Then let

$$\mathbb{C}[X] = \mathbb{C}\{X^\lambda \mid \lambda \in P\} \qquad \text{with} \quad X^\lambda X^\mu = X^{\lambda+\mu}. \tag{8.2}$$

The Weyl group of $\mathfrak{g}$ acts on $\mathbb{C}[X]$ by $wX\lambda = X^{w\lambda}$ for $w \in W$.

Let V be a finite-dimensional $\mathfrak{g}$-module. The *character* associated to V is the element of $\mathbb{C}[X]$ given by

$$\mathrm{ch}(V) = \sum_{\lambda \in P} \dim(V_\lambda) X^\lambda \tag{8.3}$$

where $V_\lambda = \{v \in V \mid hv = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}$ is the λ-weight space of V. So $\mathrm{ch}(V)$ encodes the dimensions of the weight spaces, and therefore the dimension of the whole module ($\mathrm{ch}(V)\big|_{X=1} = \dim(V)$). Further, the action of W on the weight spaces preserves dimension (by Proposition 5.4(c)), so $\mathrm{ch}(V)$ is symmetric with respect to the action of W, meaning that for all $w \in W$

$$w \cdot \mathrm{ch}(V) = \sum_{\lambda \in P} \dim(V_\lambda) X^{w(\lambda)} = \sum_{\lambda \in P} \dim(V_{w^{-1}(\lambda)}) X^\lambda = \sum_{\lambda \in P} \dim(V_\lambda) X^\lambda = \mathrm{ch}(V). \tag{8.4}$$

Proposition 8.3 ([Ser, §VII.7, Prop 5])**.** *Let* V, V' *be finite-dimensional* $\mathfrak{g}$*-modules with* $\mathrm{ch}(V)$ *as in* (8.3).

(1) The character $\mathrm{ch}(V)$ *is symmetric with respect to the action of* W*, so*

$$\mathrm{ch}(V) \in \mathbb{C}[X]^W = \{f \in \mathbb{C}[X] \mid wf = f\}.$$

(2) One has

$$\mathrm{ch}(V \oplus V') = \mathrm{ch}(V) + \mathrm{ch}(V') \quad \textit{and} \quad \mathrm{ch}(V \otimes V') = \mathrm{ch}(V)\mathrm{ch}(V')$$

(3) The modules V *and* V' *are isomorphic if and only if* $\mathrm{ch}(V) = \mathrm{ch}(V')$.

Proof. Part (1) follows from Proposition 5.4(c) and (8.4). Part (2) follows immediately for the sum rule, and by a similar analysis as we did for $\mathfrak{sl}_2$ in (3.1) for the product rule. For (3), $\mathrm{ch}(V)$ tells you the non-zero weight spaces of V as well as the dimension. So if $\mathrm{ch}(V) = \mathrm{ch}(V')$, then V and V' have the same set of weights with the same multiplicities and the same overall dimension. SO this can be proved by induction on the dimension of V and V'. First, $\mathrm{ch}(V) = 0$ if and only if $V = 0$. Next, since V is finite-dimensional, the set $P_V = \{\lambda \in P \mid V_\lambda \neq 0\}$ of weights occurring in V is finite. So there's some $\lambda \in P_V$ with $\lambda + \alpha \notin P_V$ for all $\alpha \in R^+$. Then any non-zero element $v_\lambda \in V_\lambda$ is primitive, and therefore generates $L(\lambda) \in V$. So $V \cong L(\lambda) \oplus V^\perp$, with $\mathrm{ch}(V^\perp) = \mathrm{ch}(V) - \mathrm{ch}(L(\lambda)$. But of course, if the characters of V and V' are the same, then there's also a non-zero primitive $v'_\lambda \in V'_\lambda$. So $V' \cong L(\lambda) \oplus V'^\perp$, which implies $\mathrm{ch}(V'^\perp) = \mathrm{ch}(V') - \mathrm{ch}(L(\lambda) = \mathrm{ch}(V) - \mathrm{ch}(L(\lambda) = \mathrm{ch}(V^\perp)$. Therefore, by induction on the dimension of V and V', $V \cong V'$.

□

One might want to study the the structure of the subalgebra of $\mathbb{C}[X]^W$ consisting of the characters of finite-dimensional representations. In fact (given here without proof), the whole of $\mathbb{C}[X]^W$ is generated by the characters corresponding to the fundamental representations. In other words, if

$$\chi_i = \mathrm{ch}(L(\omega_i)), \qquad \text{then} \qquad \mathbb{C}[X]^W = \mathbb{C}[\chi_1, \ldots, \chi_r].$$

In case you have heard of such things, this means that the map

$$\mathrm{ch} : \{\text{ finite dimensional } \mathfrak{g}\text{-modules }\} \to \mathbb{C}[X]^W$$

induces an isomorphism from the *Grothendieck group* on finite-dimensional $\mathfrak{g}$-modules onto $\mathbb{C}[X]^W$.

With $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha = \sum_{i=1}^r \omega_i$ as in (7.1) and $\det : W \to \{\pm 1\}$ as in (7.3), define the *Weyl denominator* as

$$a_\rho = \sum_{w \in W} \det(w) X^{w(\rho)} = \prod_{\alpha \in R^+} (X^{\frac{1}{2}\alpha} - X^{-\frac{1}{2}\alpha}). \tag{8.5}$$

The equality between the sum and product formulas there takes a little work. However, for example, when $\mathfrak{g} = \mathfrak{sl}_2$, $B = R^+ = \{\alpha = \varepsilon_1 - \varepsilon_2\}$, so that $W = \langle s_\alpha \rangle = \mathbb{Z}_2$ and $\rho = \frac{1}{2}\alpha$. Then

$$\begin{aligned} a_\rho &= \sum_{w \in W} \det(w) X^{w(\rho)} = X^{\frac{1}{2}\alpha} - X^{s_\alpha(\frac{1}{2}\alpha)} = X^{\frac{1}{2}\alpha} - X^{\frac{1}{2}\alpha - \alpha} \\ &= X^{\frac{1}{2}\alpha} - X^{-\frac{1}{2}\alpha} = \prod_{\alpha \in R^+} (X^{\frac{1}{2}\alpha} - X^{-\frac{1}{2}\alpha}). \end{aligned}$$

When $\mathfrak{g} = \mathfrak{sl}_3$, with $\beta_1 = \varepsilon_1 - \varepsilon_2$, $\beta_2 = \varepsilon_2 - \varepsilon_3$, you'll be asked to show on the homework that

$$a_\rho = X^{\beta_1 + \beta_2} - X^{\beta_2} - X^{\beta_1} + X^{-\beta_2} + X^{-\beta_1} - X^{-\beta_1 - \beta_2} \tag{8.6}$$

using either definition.

Theorem 8.4 (Weyl character formula)**.** *With $\lambda \in P^+$ a dominant integral weight, and $L(\lambda)$ the highest weight module of weight λ, we have*

$$\mathrm{ch}(L(\lambda)) = \frac{a_{\lambda+\rho}}{a_\rho}, \qquad \text{where} \quad a_{\lambda+\rho} = \sum_{w \in W} \det(w) X^{w(\lambda+\rho)}.$$

Algebraic proofs can be found in [Bou, §VIII.9, Thm 1] and [Hum, §24.3. Thm 1]. The original proof is due to Weyl (1926), and used theory of compact groups.

Again, let's convince ourselves with a couple of examples.

Example. *For any Lie algebra, $\lambda = 0$ indexes the trivial module. Then*

$$\mathrm{ch}(L(0)) = \frac{a_{0+\rho}}{a_\rho} = 1,$$

saying $L(0)$ has one weight space of weight 0 with multiplicity 1, just as expected.

Example. *When $\mathfrak{g} = \mathfrak{sl}_2$, the finite-dimensional representations are $L(d\alpha/2)$, indexed by non-negative integers d (the fundamental weight is $\omega = \frac{1}{2}\alpha$, so $P^+ = \mathbb{Z}_{\geq 0}\{\alpha/2\}$). Then since $\rho = \frac{1}{2}\alpha$, we have*

$$a_\rho = X^{\alpha/2} - X^{-\alpha/2} \quad \text{and} \quad a_{d\alpha/2+\rho} = X^{(d+1)\alpha/2} - X^{-(d+1)\alpha/2}$$

and so using the fact that

$$\frac{x^n - x^{-n}}{x - x^{-1}} = x^{n-1} + x^{n-3} + \cdots + x^{-n+3} + x^{-n+1} \quad \text{with } x = X^{\alpha/2},$$

we have

$$\mathrm{ch}(L(\lambda)) = \frac{X^{(d+1)\alpha/2} - X^{-(d+1)\alpha/2}}{X^{\alpha/2} - X^{-\alpha/2}} = X^{d\alpha/2} + X^{(d-2)\alpha/2} + \cdots + X^{-(d-2)\alpha/2} + X^{-d\alpha/2}.$$

From this character, we can read that the highest weight module indexed by d has weights $d, d-2, \dots, -d+2, -d$, all with multiplicity 1, which is exactly what we expected.

Example. *When $\mathfrak{g} = \mathfrak{sl}_3$, let's return to Example 8.1, where $\lambda = \beta_1 + \beta_2 = \rho = \varepsilon_1 - \varepsilon_3$. Recall that $\rho = \beta_1 + \beta_2 = \lambda$, and*

$$a_\rho = (X^{\frac{1}{2}\beta_1} - X^{-\frac{1}{2}\beta_1})(X^{\frac{1}{2}\beta_2} - X^{-\frac{1}{2}\beta_2})(X^{\frac{1}{2}(\beta_1+\beta_2)} - X^{-\frac{1}{2}(\beta_1+\beta_2)}).$$

For the numerator of $\mathrm{ch}(V)$, note that for this particular example, $a_{\lambda+\rho} = a_{2\rho}$ and so

$$a_{\lambda+\rho} = a_{2\rho} = \sum_{w \in W} \det(w) X^{w(2\rho)} = \sum_{w \in W} \det(w) X^{2w(\rho)}.$$

Therefore, we can cheat and substitute X^2 for X in (8.5) to get

$$a_{\lambda+\rho} = (X^{\beta_1} - X^{-\beta_1})(X^{\beta_2} - X^{-\beta_2})(X^{(\beta_1+\beta_2)} - X^{-(\beta_1+\beta_2)})$$

so that $(x^2 - x^{-2})/(x - x^{-1}) = x + x^{-1}$ gives

$$\begin{aligned}
\mathrm{ch}(L(\rho)) &= \left(\frac{X^{\beta_1} - X^{-\beta_1}}{X^{\frac{1}{2}\beta_1} - X^{-\frac{1}{2}\beta_1}}\right)\left(\frac{X^{\beta_2} - X^{-\beta_2}}{X^{\frac{1}{2}\beta_2} - X^{-\frac{1}{2}\beta_2}}\right)\left(\frac{X^{(\beta_1+\beta_2)} - X^{-(\beta_1+\beta_2)}}{X^{\frac{1}{2}(\beta_1+\beta_2)} - X^{-\frac{1}{2}(\beta_1+\beta_2)}}\right) \\
&= (X^{\frac{1}{2}\beta_1} + X^{-\frac{1}{2}\beta_1})(X^{\frac{1}{2}\beta_2} + X^{-\frac{1}{2}\beta_2})(X^{\frac{1}{2}(\beta_1+\beta_2)} + X^{-\frac{1}{2}(\beta_1+\beta_2)}) \\
&= X^{\beta_1+\beta_2} + X^{\beta_1} + X^{\beta_2} + 2 + + X^{-\beta_2} + X^{-\beta_1} + X^{-(\beta_1+\beta_2)}.
\end{aligned}$$

This says that $L(\rho)$ has seven weight spaces, six of dimension 1 corresponding to $\alpha \in R$, and one of dimension 2, corresponding to the weight 0. This is exactly what we got in Example 8.1.

Remark 8.5. In type A, the entire theory worked out for type A in [Mac, §I.3]. In particular, $\mathrm{ch}(L(\lambda))$ is the *Schur function* corresponding to the dominant integral weight λ. When hear the phrase *Schur positivity*, people are trying to figure out if some symmetric function they're interested in can be expressed as a positive integer combination of Schur functions, since any such function is secretly a character associated to a $\mathfrak{sl}_n$-module. See [Mac] for a full discussion of Schur functions and other great generalizations.

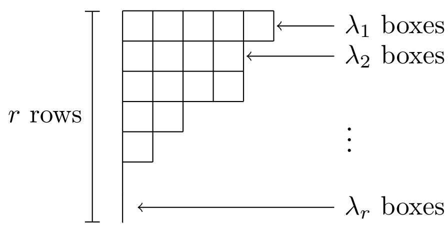
8.2.1. *Trick for type A_r: Partitions, compositions, and Young tableaux.* Recall, for $\mathfrak{g} = A_r$,

$$P^+ = \mathbb{Z}_{\geq 0}\Omega = \{\lambda_1\varepsilon_1 + \cdots + \lambda_r\varepsilon_r - \frac{|\lambda|}{r+1}\varepsilon_1 + \cdots + \varepsilon_{r+1} \mid **\}$$

where

$$** = \begin{Bmatrix} \lambda_i \in \mathbb{Z}_{\geq 0}, & |\lambda| = \lambda_1 + \cdots + \lambda_r \\ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0 \end{Bmatrix}.$$

So P^+ is in bijection with integer partitions of length less than or equal to r. We can draw integer partitions as $|\lambda|$ boxes piled up and left into a corner, with λ_i boxes in the ith row:



A *composition* is a partition without the condition that $\lambda_i \geq \lambda_{i+1}$, and can also be drawn as a left-justified box arrangement with λ_i boxes in the ith row. Define the *weight* of a composition as the collection of integers with

$$\lambda_1 \text{ 1's}, \quad \lambda_2 \text{ 2's}, \cdots, \lambda_r\ r\text{'s}.$$

You can visualize this as the filling of a composition λ with 1's in the 1st row, 2's in the 2nd row, and so on:

$$\begin{ytableau} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 \\ 5 \end{ytableau} \qquad \begin{ytableau} 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \\ 4 & 4 \\ \none \\ 6 & 6 & 6 & 6 \end{ytableau}$$

Let λ be a partition and μ a composition with $|\lambda| = |\mu|$.
A *semistandard tableau* or *filling* of shape λ and weight μ is a filling of the boxes in λ with the integers in $\mathrm{wt}(\mu)$ such that rows weakly increase and columns strictly increase.

For example, there are 2 semistandard fillings of $\ydiagram{3,1}$ with weight $\ydiagram{2,1,1}$:

$$\begin{ytableau} 1 & 1 & 2 \\ 3 \end{ytableau} \qquad \begin{ytableau} 1 & 1 & 3 \\ 3 \end{ytableau}$$

but there are no semistandard fillings of $\ydiagram{2,1,1}$ with weight $\ydiagram{3,1}$.

Let λ be a partition with r or fewer parts, abusing notation let $L(\lambda)$ be the corresponding module. It turns out that the weights in P_λ can all be expressed as compositions of $|\lambda|$ of length $\leq r+1$, and the dimension of the weight space (corresponding to composition μ) is equal to the number semistandard fillings of λ with weight μ.

Example. *Returning again to $\mathfrak{g} = \mathfrak{sl}_3$ and $\lambda = \beta_1 + \beta_2 = \omega_1 + \omega_2 = \rho = 2\varepsilon_1 + \varepsilon_2 - \frac{3}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$, so λ corresponds to the partition $(2, 1, 0) = $ ⊞. Let $\gamma = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$. We can rewrite all $\mu \in P_\lambda$ uniquely as*

$$\mu_1\varepsilon_1 + \mu_2\varepsilon_2 + \mu_3\varepsilon_3 - \frac{|\mu|}{3}$$

with $\mu_1 + \mu_2 + \mu_3 = \lambda_1 + \lambda_2 = 2 + 1 = 3$. Then this correspondence is given by the following table.

weight μ in P_ρ	*rewriting μ*	*composition (μ_1, μ_2, μ_3)*	*fillings of λ*
$\varepsilon_1 - \varepsilon_3$	$2\varepsilon_1 + \varepsilon_2 - \gamma$	$(2, 1, 0)$	$\begin{array}{ll} 1 & 1 \\ 2 & \end{array}$
$\varepsilon_1 - \varepsilon_2$	$2\varepsilon_1 + \varepsilon_3 - \gamma$	$(2, 0, 1)$	$\begin{array}{ll} 1 & 1 \\ 3 & \end{array}$
$\varepsilon_2 - \varepsilon_3$	$2\varepsilon_2 + \varepsilon_3 - \gamma$	$(0, 2, 1)$	$\begin{array}{ll} 2 & 2 \\ 3 & \end{array}$
$\varepsilon_3 - \varepsilon_1$	$\varepsilon_2 + 2\varepsilon_3 - \gamma$	$(0, 1, 2)$	$\begin{array}{ll} 2 & 3 \\ 3 & \end{array}$
$\varepsilon_2 - \varepsilon_1$	$2\varepsilon_2 + \varepsilon_3 - \gamma$	$(0, 2, 1)$	$\begin{array}{ll} 2 & 2 \\ 3 & \end{array}$
$\varepsilon_3 - \varepsilon_2$	$\varepsilon_1 + 2\varepsilon_3 - \gamma$	$(1, 0, 2)$	$\begin{array}{ll} 1 & 3 \\ 3 & \end{array}$
0	$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \gamma$	$(1, 1, 1)$	$\begin{array}{ll} 1 & 2 \\ 3 & \end{array}$ $\begin{array}{ll} 1 & 3 \\ 2 & \end{array}$

The connection to symmetric functions in $r + 1$ variables is to let

$$x_i = X^{\varepsilon_i - \frac{1}{r+1}(\varepsilon_1 + \cdots + \varepsilon_{r+1})}, \qquad i = 1, \ldots, r + 1.$$

Then $\mathrm{ch}(\lambda)$ turns out to be the sum over all compositions μ of the same size a λ with $r+1$ (possibly trivial parts) of

$$\#\{\text{s.s. fillings of } \lambda \text{ with weights } \mu\ \} x_1^{\mu_1} \cdots x_{r+1}^{\mu_{r+1}}.$$

8.3. **Path model.** The main references on this section are [Li95] where the paths and root operators are developed, and [Ra06, §5] where a survey of crystals is given along with some connections to Hecke algebras.

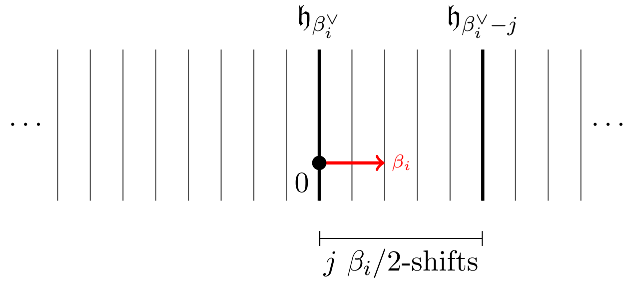
A *path* is a piecewise linear continuous (non-pathological) map

$$p : [0, 1] \to \mathfrak{h}_\mathbb{R}^* \qquad \text{with} \qquad p(0) = 0 \text{ and } p(1) \in P.$$

We put an equivalence on paths given by $p_1 \sim p_2$ if there's a continuous non-decreasing bijective map $\phi : [0, 1] \to [0, 1]$ with $p_1 = p_2 \circ \phi$.

Pick a simple root $\beta_i \in B$, and define simple root operators e_i and f_i as follows. Abusing notation, with $a \in \mathbb{R}$ and let $\mathfrak{h}_{\beta_i - a}$ be the hyperplane parallel to $\mathfrak{h}_{\beta_i}$ shifted by $(a/2)\alpha$, so that

$$\mathfrak{h}_{\beta_i^\vee - a} = \{\lambda \in \mathfrak{h}_\mathbb{R}^* \mid \langle \lambda, \beta_i^\vee - a\rangle = 0\} = \{\lambda \in \mathfrak{h}_\mathbb{R}^* \mid \langle \lambda, \beta_i^\vee \rangle = a\}.$$



Fix a path p, and let $a = \min_{0 \le t \le 1} \langle p(t), \beta_i^\vee \rangle$. Consider the region contained between $\mathfrak{h}_{\beta_i^\vee - a}$ and $\mathfrak{h}_{\beta_i^\vee - (a+1)}$. Highlight parts of the paths in this region as follows.

Let t_L be maximal such that $\langle p(t_L), \beta_i^\vee \rangle = a$ (the last place where the path hits $\mathfrak{h}_{\beta_i^\vee - a}$). If $\langle p(1), \beta_i^\vee \rangle \ge a+1$, then let t_R be minimal such that $\langle p(t_R), \beta_i^\vee \rangle \ge a+1$ for all $[t_R, 1]$ (the first point where p crosses $\mathfrak{h}_{\beta_i^\vee - (a+1)}$. Then fix a finite partition of $[0,1]$ given by

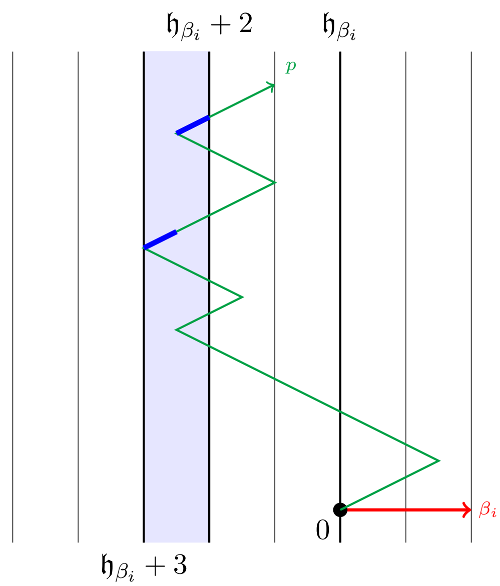
$$t_L = t_0 < t_1 < \cdots < t_m = t_R$$

such that either

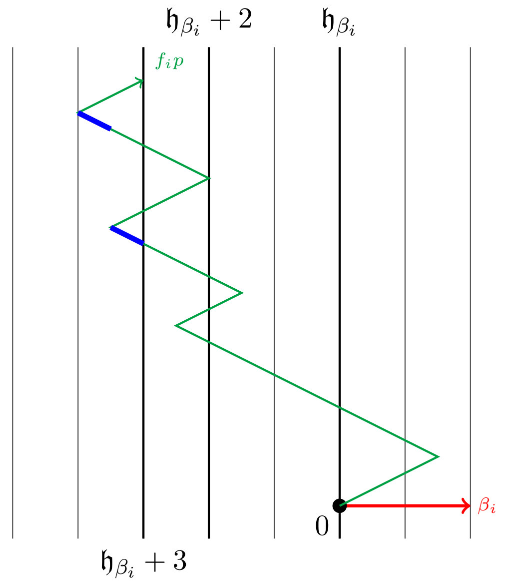
(1) $\langle p(t_j), \beta_i^\vee \rangle = \langle p(t_{j+1}), \beta_i^\vee \rangle$ and $\langle p(t), \beta_i^\vee \rangle \ge \langle p(t_j), \beta_i^\vee \rangle$ for $t \in [t_j, t_{j+1}]$ (p starts on the hyperplane intersecting $p(t_j)$, heads to the positive side and doubles back to that same hyperplane), or

(2) $\langle p(t), \beta_i^\vee \rangle$ is strictly increasing for $t \in [t_j, t_{j+1}]$ and $\langle p(t), \beta_i^\vee \rangle \ge \langle p(t_{j+1}), \beta_i^\vee \rangle$ for all $t \ge t_{j+1}$ (p heads in the positive direction, and there's nothing later on the path further to the negative).

Highlight all segments of the path p on intervals of the second kind. For example,



If $\langle p(1), \beta_i^\vee \rangle < a+1$, the operator f_i acts by 0. Otherwise, then operator f_i reflects each highlighted segment from the positive to the negative side the hyperplane $\mathfrak{h}_{\beta_i - \langle p(s_j), \beta_i^\vee \rangle}$, dragging the rest of the path with it:



Note that the result move the end of the path by $-\beta_i$. The operators e_i reverses this operation, i.e. it is defined by $e_i f_i = 1$ (whenever f_i does not act by 0) and $f_i e_i = 1$ (whenever e_i does not act by 0). It still uses the part of the part furthest in the $-\beta_i$-direction, but reflecting from the negative side to the positive side of a hyperplane. So

$$f_i p = 0 \text{ or } (f_i p)(1) = p(1) - \beta_i \qquad \text{and} \qquad e_i p = 0 \text{ or } (e_i p)(1) = p(1) + \beta_i.$$

A *crystal* $\mathcal{B}$ is a set of paths which is closed under the action of root operators $\{e_i, f_i \mid \beta_i \in B\}$. Let $\mathcal{B}(p)$ be the minimal crystal containing p.

Example. *Let $\mathfrak{g} = \mathfrak{sl}_3$ with base $B = \{\beta_1 = \varepsilon_1 - \varepsilon_2, \beta_2 = \varepsilon_2 - \varepsilon_3\}$. Let p be the straight-line path from 0 to $\lambda = \varepsilon_1 - \varepsilon_3$. Then $\mathcal{B}(p)$, together with its actions by e_1, e_2, f_1, f_2, is given in Figure 2.*

We say $\mathcal{B}$ and $\mathcal{B}'$ are *isomorphic crystals* if there is a bijection $\phi : \mathcal{B}(p) \to \mathcal{B}(p')$ with $f_i\phi(q) = \phi(f_i q)$ and $e_i\phi(q) = \phi(e_i q)$ for all $q \in \mathcal{B}(p)$ and simple root operators f_i, e_i. The *path model*crystal graph is the graph with

$$\text{vertices } p \in \mathcal{B} \qquad \text{and} \qquad \text{labeled edges } p \xrightarrow{i} f_i p.$$

See, for example, Figure 3. Two crystals are isomorphic if and only if they have isomorphic graphs (with the same labelings of edges).

Note that since $\rho = \sum_i \omega_i$, by pulling the (open) fundamental chamber C back by ρ, $C - \rho$ contains the closed chamber $\bar{C}$, but not any of the walls $\mathfrak{h}_{\beta_i+1}$. In fact,

$$P^{++} \to P^{+} \qquad \text{defined by} \qquad \lambda \mapsto \lambda - \rho$$

is a bijection.

A *highest weight path* is a path p satisfying

$$e_i p = 0 \qquad \text{for all } i = 1, \ldots, r.$$

For e_i to act by 0 means that $\langle p(t), \beta_i^\vee \rangle > -1$ for all t and i. So a path is highest weight if and only if

$$p(1) \in P^{+} \qquad \text{and} \qquad p(t) \in C - \rho \text{ for all } t \in [0, 1].$$

The *weight* of any path p is $\mathrm{wt}(p) = p(1)$.

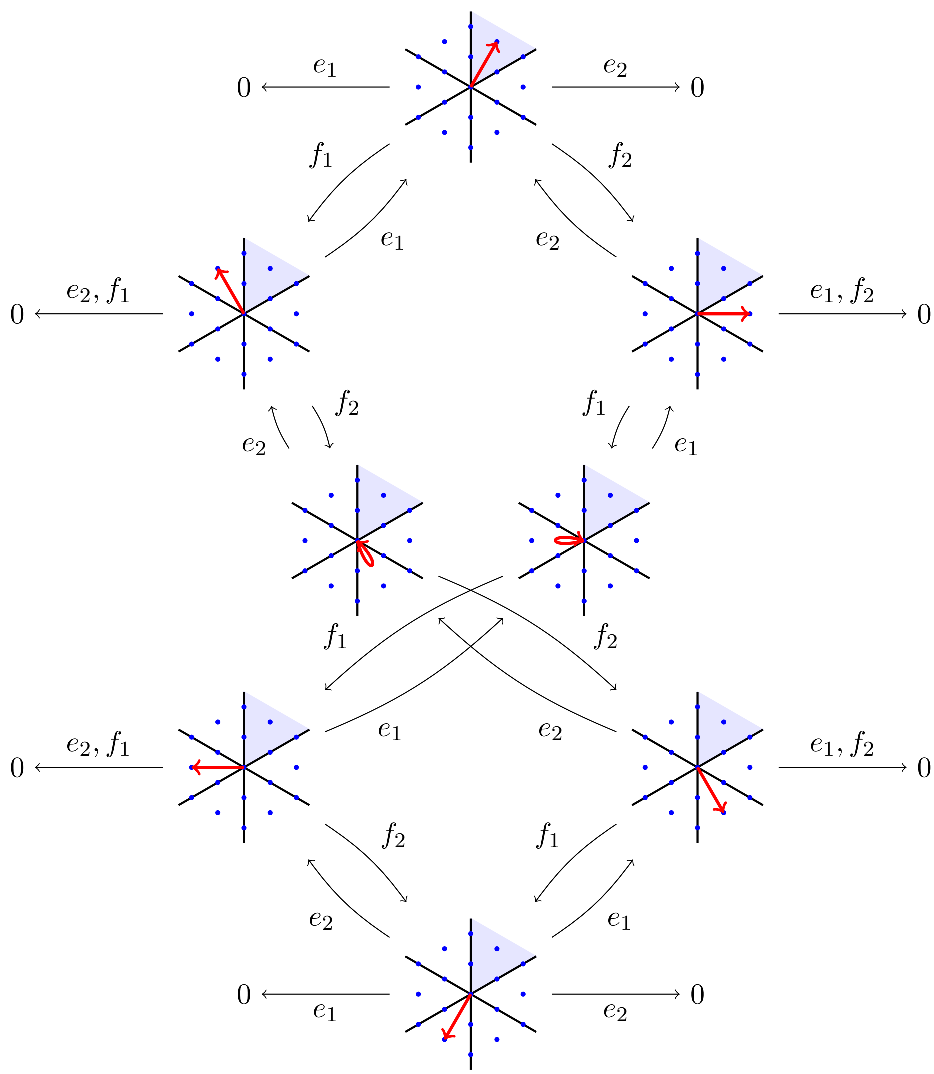
Proposition 8.6. *Let p and p' be highest weight paths of the same weight. Then the crystals generated p and p' are isomorphic.*

So without ambiguity, for $\lambda \in P^+$, define $\mathcal{B}(\lambda)$ as the crystal generated any fixed highest weight path p_λ^+ of weight λ. In Example 8.3, we computed $\mathcal{B}(\rho)$. The same crystal with a different highest weight path is in Figure 3.

The *character of a crystal* is

$$\mathrm{ch}(\mathcal{B}) = \sum_{p \in \mathcal{B}} X^{\mathrm{wt}(p)}. \tag{8.7}$$

FIGURE 2. The crystal generated by the straight-line path to ρ for $\mathfrak{sl}_3$.



Theorem 8.7. *For $\lambda \in P^+$,*
$$\mathrm{ch}(\mathcal{B}(\lambda)) = \mathrm{ch}(L(\lambda)).$$

So, for example, Example 8.3 shows once again that the weight space of weight 0 in $L(\rho)$ has dimension 2.

Remark 8.8. One thing that's more exciting about the path model than methods in the previous two sections is that we don't just have a count of the multiplicities of weights, but we have a set indexing the individual dimensions of the weight spaces of any finite-dimensional $\mathfrak{g}$-module. In other words, for any finite-dimensional $\mathfrak{g}$-module $V = \sum_{\lambda \in \hat{V}} L(\lambda)$, there is a a weight basis of V indexed by paths in $p \in \sqcup_{\lambda \in \hat{V}} \mathcal{B}(\lambda)$, where if v_p is the basis element of V indexed by path p, v_P has weight $\mathrm{wt}(p)$.

The action is a little trickier, though. In an ideal world, we would hope that since f_i changes the weight of p by $-\beta_i$, maybe we would have relations like $y_{\beta_i} v_p = v_{f_i p}$ and $x_{\beta_i} v_p = v_{e_i p}$. However, this action would not satisfy the bracket relation $[x_{\beta_i}, y_{\beta_i}] = h_{\beta_i^\vee}$. There is an initial attempt to deal with this in [Li95, §2.1], where they build operators for each simple root that form an $\mathfrak{sl}_2$-triple. However, the various $\mathfrak{sl}_2$-triples for various β_i's do not interact properly with each other (they don't satisfy what are called the Chevalley-Serre relations, like $[h_\alpha, x_\beta] = \langle \alpha, \beta \rangle x_\alpha$ and $[x_\alpha, y_\beta] = \delta_{\alpha,\beta} h_{\alpha^\vee}$). There are proper normalizations from the Lie algebras side, which fall into the study of crystal bases.

Proposition 8.9. *Let $\mathcal{B}, \mathcal{B}'$ be finite crystals.*

(1) $\mathrm{ch}(\mathcal{B}) = \mathrm{ch}(\mathcal{B}')$ *if and only if* $\mathcal{B} \cong \mathcal{B}'$.

(2) The union $\mathcal{B} \sqcup \mathcal{B}'$ *is a crystal, and*
$$\mathrm{ch}(\mathcal{B} \sqcup \mathcal{B}') = \mathrm{ch}(\mathcal{B}) + \mathrm{ch}(\mathcal{B}').$$

(3)
$$\mathrm{ch}(\mathcal{B}) = \sum_{\substack{p \in \mathcal{B} \\ p \text{ is highest weight}}} \mathrm{ch}(\mathcal{B}(\mathrm{wt}(p))).$$

It begins to make sense now to intro duct the notation $\mathcal{B}(V)$ to denote a crystal associated to a finite-dimensional module V. Namely, if $V = \sum_{\lambda \in \hat{V}} L(\lambda)$, then $\mathcal{B}(V) = \bigsqcup_{\lambda \in \hat{V}} \mathcal{B}(\lambda)$.

8.3.1. *Tensor product decomposition.* The concatenation of two paths p, p' is defined by
$$pp' = \begin{cases} p(2t) & 0 \le t \le 1/2, \\ p(1) + p'(2(t - 1/2)) & 1/2 \le t \le 1. \end{cases}$$
Pictorially, think of sticking p' onto the end of p (like vector addition). Note that $\mathrm{wt}(pp') = \mathrm{wt}(p) + \mathrm{wt}(p')$.

Theorem 8.10.

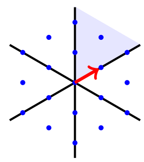
(1) For finite-dimensional $\mathfrak{g}$-modules V, V',
$$\mathcal{B}(V \otimes V') = \{pp' \mid p \in \mathcal{B}(V), p' \in \mathcal{B}(V')\}.$$

(2) With $\lambda, \mu \in P^+$, and p_λ^+ highest weight in $\mathcal{B}(\lambda)$,
$$\mathrm{ch}(L(\lambda) \otimes L(\mu)) = \sum_{\substack{q \in \mathcal{B}(\mu) \\ p_\lambda^+ q \text{ highest weight}}} \mathrm{ch}(L(\lambda + \mathrm{wt}(q))).$$

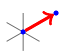
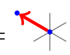
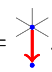
Type A_r. First, let's return to the example where $\mathfrak{g} = \mathfrak{sl}_3$. Recall the fundamental weights with resect to the base $B = \{\beta_1 = \varepsilon_1 - \varepsilon_2, \beta_2 = \varepsilon_2 - \varepsilon_3\}$ are

$$\omega_1 = \varepsilon_1 - \frac{1}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3), \quad \text{and} \quad \omega_2 = \varepsilon_1 + \varepsilon_2 - \frac{2}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3).$$

The crystal $\mathcal{B}(\omega_1)$ is generated by highest weight path

$p^+_{\omega_1}$:  ,

and contains the three paths

$p_1 = p^+_{\omega_1} =$  $p_2 = f_1 p^+_{\omega_1} =$  $p_3 = f_2 f_1 p^+_{\omega_1} =$  .

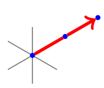
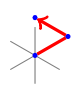
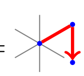
The weights of these paths are

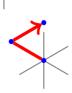
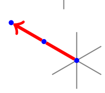
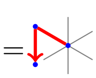
$$\begin{aligned}
\mathrm{wt}(p_1) &= \varepsilon_1 - \frac{1}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) = \omega_1, \\
\mathrm{wt}(p_2) &= \varepsilon_2 - \frac{1}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) = \omega_2 - \omega_1, \\
\mathrm{wt}(p_1) &= \varepsilon_3 - \frac{1}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) = \omega_3 - \omega_2.
\end{aligned}$$

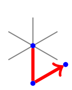
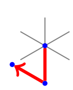
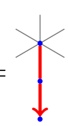
where $\omega_3 = 0 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \frac{3}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$. For general r, we saw on the homework that the standard representation of A_r is $L(\omega_1)$, and now it's not difficult to compute that weights in $L(\omega_1)$ are exactly analogous to the $r = 2$ case, namely,

$$P_{\omega_1} = \{\omega_i - \omega_{i-1} = \varepsilon_i - \frac{1}{r+1}\sum_{i=1}^{r} \varepsilon_i \mid i = 1, \dots, r+1, \omega_0 = \omega_{r+1} = 0\}.$$

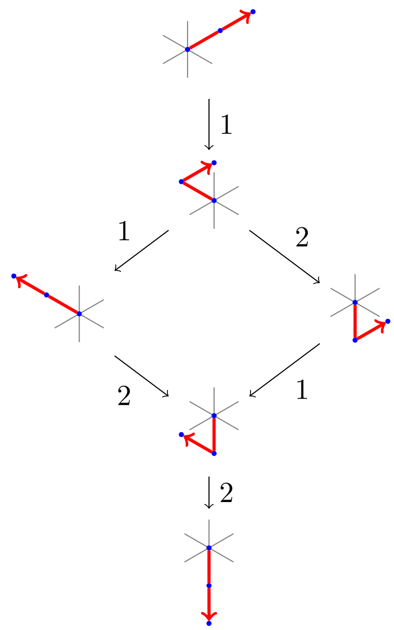
Back in the case where $r = 2$, the crystal for $\mathcal{B}(L(\omega_1) \otimes L(\omega_1))$ is the set containing

$p_1^2 =$  , $p_1 p_2 =$  , $p_1 p_3 =$  ,

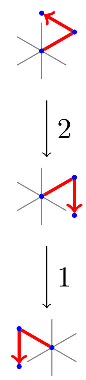
$p_2 p_1 =$  , $p_2^2 =$  , $p_2 p_3 =$  ,

$p_3 p_1 =$  , $p_3 p_2 =$  , $p_3^2 =$  .

The two of these that are highest weight are p_1^2, which has weight $2\omega_1$, and $p_1 p_2$, which has weight ω_2. This is reflected in the fact that the crystal graph for $\mathcal{B}(L(\omega_1) \otimes L(\omega_1))$ has two connected components:



and



So

$$\mathrm{ch}(L(\omega_1) \otimes L(\omega_1)) = \mathrm{ch}(L(2\omega_1)) + \mathrm{ch}(L(\omega_2)),$$

implying

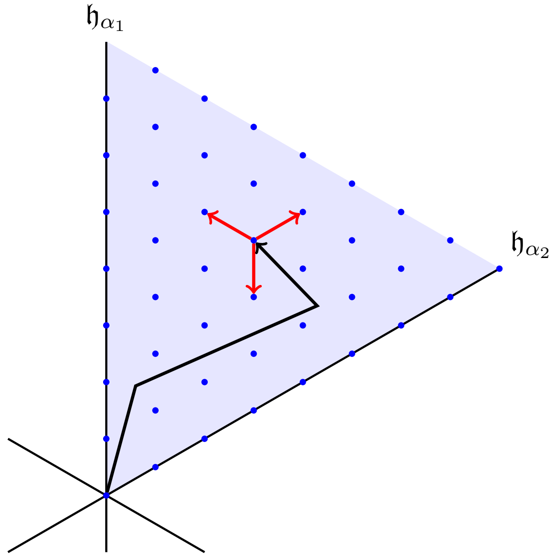
$$L(\omega_1) \otimes L(\omega_1) \cong L(2\omega_1) \oplus L(\omega_2).$$

Stepping back, since p_1 is the highest weight path of $\mathcal{B}(\omega_1)$, the highest weight paths in this tensor product are those for which $p_1 p_i$ are in $C - \rho$. More to the point, since concatenation by p_i walks from one integral weight to one to the six nearest integral weights, $p_1 p_i$ is highest weight exactly when $\mathrm{wt}(p_1 p_i) \in P^+$.

In general, for $\lambda \in P^+$, we have that

$$L(\lambda) \otimes L(\omega_1) = \bigoplus_{\substack{i=1,\ldots,r+1 \\ \lambda+\omega_i-\omega_{i-1} \in P^+}} L(\lambda + \omega_i - \omega_{i-1}),$$

which has exactly $r+1$ terms when $\lambda \in P^{++}$.



In the language of partitions, since adding $\varepsilon_i - \frac{1}{r+1}\sum_{i=1}^{r+1}\varepsilon_i$ to λ is the same as adding a box to the partition corresponding to λ in the i's row (where adding a box in the $r+1$ row is equivalent to subtracting the first column). The result yields a dominant weight exactly when adding a box yields a partition (rather than a non-partition composition). Finally, ω_1 corresponds to the partition of 1, a single box. So put in this combinatorial language

$$L(\lambda) \otimes L(\square) = \sum_{\mu \in \lambda^+} L(\mu) \tag{8.8}$$

where

$$\lambda^+ = \{ \text{ partitions obtained from } \lambda \text{ by adding a box } \}.$$

In our example above, we saw that

$$L(\square) \otimes L(\square) = L(\square\square) \oplus L\left(\begin{smallmatrix}\square\\ \square\end{smallmatrix}\right).$$

since $\omega_2 = \begin{smallmatrix}\square\\ \square\end{smallmatrix}$ and $2\omega_1 = \square\square$.

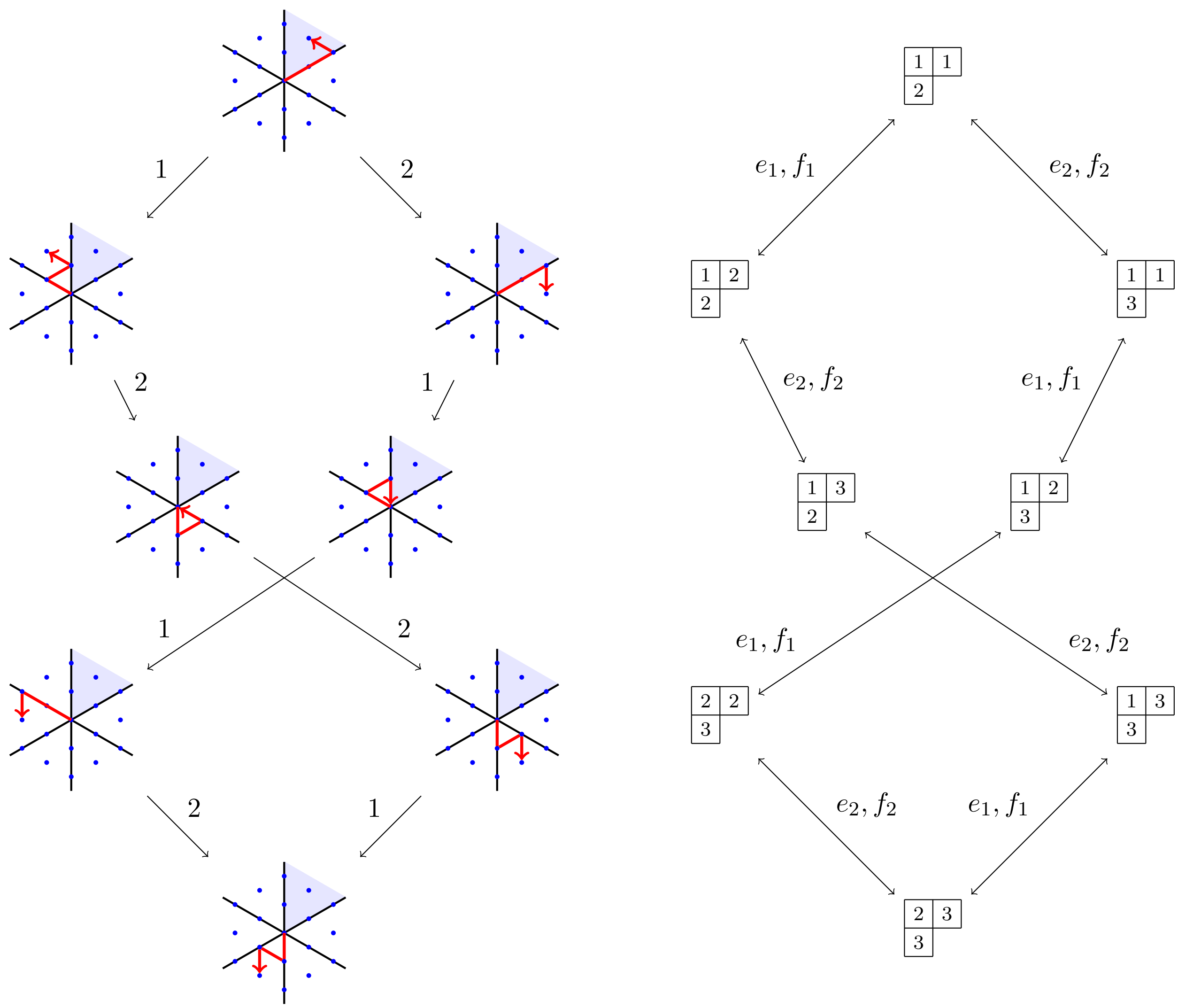
To connect back to the tableaux in Section 8.2.1, let's start to use these more discrete paths to generate crystals. Keeping with p_i being the straight-line path to $\varepsilon_i - \frac{1}{r+1}(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{r+1})$, we have $p^+_{\omega_1} = p_1$, and $\mathcal{B}(\omega_1) = \{p_i \mid i = 1, \ldots, r+1\}$. With $\lambda \in P^+$, instead of starting with the straight line path to λ, rewrite

$$\lambda = \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 + \cdots + \lambda_r\varepsilon_r - \frac{|\lambda|}{r+1}(\varepsilon_1 + \cdots + \varepsilon_{r+1})$$

(where $|\lambda| = \lambda_1 + \cdots + \lambda_r$ is not the same thing as $||\lambda||$). Then take the path which is the concatenation of paths p_i according to

$$p = \underbrace{p_1 p_1 \cdots p_1}_{\lambda_1}\underbrace{p_2 p_2 \cdots p_2}_{\lambda_2} \cdots = p_1^{\lambda_1} p_1^{\lambda_1} \cdots p_r^{\lambda_r}.$$

Figure 3. The $\mathfrak{sl}_3$ crystal graph generated corresponding to ρ. The left is generated by the path to ρ given by $p = p_1^{\lambda_1} p_2^{\lambda_2} \dots p_4^{\lambda_r}$. The right shows the corresponding fillings of the partition corresponding to ρ. Labeled edges indicate $p \xrightarrow{i} f_i p$; missing edges indicate e_i or f_i acts by 0.



Since $\mathrm{wt}(p_i) = \omega_i - \omega_{i-1}$ and $\lambda_i \le \lambda_{i-1}$, we have $p = p_1^{\lambda_1} p_1^{\lambda_1} \cdots p_r^{\lambda_r} \in C - \rho$ (in fact, $p \in \bar{C}$). In fact, for any path constructed as

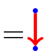
$$p = p_{i_1} p_{i_2} \cdots p_{i_n},$$

p is in $C - \rho$ if and only if every initial path $p_{i_1} \cdots p_{i_j}$ has weight

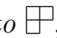
$$\mathrm{wt}(p_{i_1} \cdots p_{i_j}) = \sum_{k=1}^{j} (\omega_{i_k} - \omega_{i_k - 1}) \in P^+$$

(if any only if it's the positive sum of ω's). Another way to put this is that if we define the *reading word* of p to be $i_1 i_2 \cdots i_n$, then p is in $C - \rho$ if any only if every initial subword $i_1 i_2 \cdots i_j$ of the reading word of p has the property that is contains more 1's than 2's, more 2's than 3's, and so on. This should sound wildly familiar if you are acquainted with the Littlewood-Richardson rule, which we'll get to momentarily.

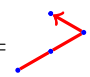
Example. *Let's return to the example where $r = 2$ and $\lambda = \rho$, and*

$p_1 =$  $\quad p_2 =$  $\quad$ *and* $\quad p_3 =$ .

Since

$$\rho = \varepsilon_1 - \varepsilon_3 = 2\varepsilon_1 + \varepsilon_2 - \frac{3}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$$ *corresponds to* ,

the highest weight path we want to start with is

$p = p_1 p_1 p_2 =$ 

Then the resulting crystal is in Figure 3.

When we construct a crystal in this way, generated by highest weight path $p = p_1^{\lambda_1} \cdots p_\ell^{\lambda_\ell}$, there is a very straightforward correspondence between the semistandard fillings of the partition λ in Section 8.2.1 and the reading words of the paths in $\mathcal{B}(p_1^{\lambda_1} \cdots p_\ell^{\lambda_\ell})$. Namely, every path in $\mathcal{B}(p_1^{\lambda_1} \cdots p_\ell^{\lambda_\ell})$ is of the form $q = p_{i_1} \cdots p_{i_n}$ with $n = |\lambda|$, and

(1) if q has weight μ, the collection of integers $i_1, i_2, \ldots, i_n$ is the weight of the composition μ; and
(2) filling λ, reading right to left, top to bottom, with the word $i_1 i_2 \cdots i_n$ yields a semistandard filling of λ with weight μ.

Moreover, this correspondence gives a bijection. The example where $\mathfrak{g} = \mathfrak{sl}_3$ and $\lambda = \rho$ is also in Figure 3.

Type C_r. In type C_r, there's a similar story. Choose the base

$$B = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{r-1} - \varepsilon_r, 2\varepsilon_r\}$$

so that the fundamental weights are given by

$$\omega_i = \varepsilon_1 + \cdots + \varepsilon_i \qquad \text{for } i = 1, \ldots, r.$$

We saw on the homework that for $r = 2$, $L(\omega_1)$ is a four-dimensional module, with all one-dimensional weight spaces corresponding to the weights

$$\{\pm\varepsilon_1, \pm\varepsilon_2\}.$$

Similar to the proof on the homework for type A_r, you can check that $L(\omega_1)$ is the standard representation for C_r in general. (In fact, that's true across types!) It has one-dimensional weight spaces of weights

$$\{\pm\varepsilon_1, \pm\varepsilon_2, \cdots, \pm\varepsilon_r\}$$

(so that it has dimension $2r$, as expected).

Similar to the type A_r case, there's also a connection to partitions in the C_r case as well. Namely,

$$P^+ = \left\{ \lambda_1\varepsilon_1 + \cdots + \lambda_r\varepsilon_r \;\middle|\; \begin{array}{c} \lambda_i \in \mathbb{Z} \\ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0 \end{array} \right\}$$

is in bijection with integer partitions of length at most r (with less work than in type A_r, even).

On the homework, you're asked to give the analogous decomposition to (8.8) for type C_r.

9. Centralizer algebras

As I hope I've conveyed, one big goal in representation theory is to know how representations decompose into irreducible components. In general, indecomposable (breaks further into direct summands) doesn't imply irreducible (contains no proper non-trivial invariant subspaces). But by definition, modules for semisimple algebras are indecomposable if and only if they are irreducible. So the trick is to figure out *how* to decompose a given module. A good reference for this section is [GW, §4].

At the level of individual modules, this can be difficult, since the decomposition might not be unique. But at the level of isotypic components, the decomposition is canonical. Namely, let $\widehat{A}$ be an indexing set for the isomorphism classes of irreducible A-modules, and for $\lambda \in \widehat{A}$, let A^λ be the A-module indexed by λ. Then for an A module M, the *isotypic component* of M corresponding to λ is

$$M^{(\lambda)} = \sum_{\substack{U \subseteq M \\ U \cong A^\lambda}} U,$$

the subspace of M generated by all submodules isomorphic to A^λ. Though the multiplicity $m_M(\lambda)$ of A^λ in M (and so in $M^{(\lambda)}$) is well-defined, the mechanical decomposition

$$M^{(\lambda)} = \bigoplus_{i=1}^{m_M(\lambda)} A^\lambda = m_M(\lambda) A^\lambda$$

is not unique. However, the decomposition

$$M = \bigoplus_{\lambda \in \widehat{M}} M^{(\lambda)} \qquad \text{where } \widehat{M} = \{\lambda \in \widehat{A} \mid M^{(\lambda)} \neq 0\},$$

is unique.

In the case where A is finite-dimensional, the decomposition of M into styptic components comes straight from Wedderburn's theorem: If A is finite-dimensional, *Wedderburn's theorem* says

$$A \cong \bigoplus_{\lambda \in \widehat{A}} \mathrm{End}(A^\lambda)$$

where $\mathrm{End}(A^\lambda)$ is the algebra of endomorphisms of the vector space A^λ (coming from the action of A on itself). So on each block $\mathrm{End}(A^\lambda)$, there is an identity operator I_λ which looks like 1 on $\mathrm{End}(A^\lambda)$ and 0 on $\mathrm{End}(A^\mu)$ for $\mu \neq \lambda$. These operators satisfy

(1) $I_\lambda^2 = I_\lambda$ (I_λ is an *idempotent*);
(2) $I_\lambda I_\mu = I_\mu I_\lambda = 0$ for $\lambda \neq 0$ (they are pairwise orthogonal);
(3) $\sum_{\lambda \in \widehat{A}} I_\lambda = 1$;
(4) $Z(A) = \mathbb{C}\{I_\lambda \mid \lambda \in \widehat{A}\}$; and
(5) the action of I_λ on any A-module M projects onto $M^{(\lambda)}$.

The I_λ's are called the *centrally primitive idempotents* of A.

In the case where A is infinite dimensional, we need a replacement for the projection operation from I_λ. To this end, let $\mathrm{Hom}(A^\lambda, M)$ be the set of A-module homomorphisms from A^λ into M. Since A^λ is simple, Schur's lemma tells us that every non-zero $\phi \in \mathrm{Hom}(A^\lambda, M)$ gives $\phi(A^\lambda) \cong A^\lambda$, and so $\phi(A^\lambda) \subseteq M^{(\lambda)}$. There's then a canonical map

$$\mathrm{Hom}(A^\lambda, M) \otimes A^\lambda \to M \quad \text{defined by} \quad \phi \otimes u \mapsto \phi(u),$$

which produces an isomorphism

$$\mathrm{Hom}(A^\lambda, M) \otimes A^\lambda \cong M^{(\lambda)}.$$

This shows that $m_M(\lambda) = \dim(\mathrm{Hom}(A^\lambda, M))$. In fact, more is true! There is a bilinear form on the vector space of A-modules given by $\langle M, N\rangle = \dim(\mathrm{Hom}(M, N))$, which is also symmetric since

$$\dim(\mathrm{Hom}(M, N)) = \sum_{\lambda \in \widehat{A}} m_M(\lambda) m_N(\lambda).$$

So, in particular,

$$m_M(\lambda) = \dim(\mathrm{Hom}(A^\lambda, M)) = \dim(\mathrm{Hom}(M, A^\lambda)).$$

Now let A be a semisimple algebra over $\mathbb{C}$ (associative with identity), and let M be an A-module. Let $\mathrm{End}(M)$ the endomorphisms of M (linear, not necessarily invertible, maps from M to M). To be rigorous, there is representation of A corresponding to its action on M determined by

$$\begin{aligned} \rho : A &\to \mathrm{End}(M) \\ a &\mapsto \rho(a) \end{aligned} \qquad \text{is defined by} \qquad a \cdot v = \rho(a)v.$$

But practically speaking, we often identify A with its image $\rho(A)$ in $\mathrm{End}(M)$ (even though its image is actually the quotient by $\ker(\rho)$). Now define the *centralizer* of A (in $\mathrm{End}(M)$) to be

$$\mathrm{End}_A(M) = \{\phi \in \mathrm{End}(M) \mid a\phi(m) = \phi(a \cdot m) \text{ for all } a \in A, m \in M\}. \tag{9.1}$$

There is a natural action of $B = \mathrm{End}_A(M)$ on $\mathrm{Hom}(A^\lambda, M)$ by

$$b \cdot \phi : v \mapsto b \cdot \phi(v), \qquad \text{for any } b \in B,\ \phi \in \mathrm{Hom}(A^\lambda, M), \text{ and } v \in A^\lambda.$$

This is well-defined since $\phi(v) \in M$ and $B \subseteq \mathrm{End}(M)$. The result is another A-module homomorphism since for any $a \in A$,

$$\begin{aligned} (b \cdot \phi)(a \cdot v) &= b \cdot (\phi(a \cdot v)) && \text{(the definition of the } B\text{-action)} \\ &= b \cdot (a \cdot \phi(v)) && (\phi \text{ is an } A\text{-module homomorphism)} \\ &= a \cdot (b \cdot \phi(v)) && \text{(the actions of } A \text{ and } B \text{ commute)} \\ &= a \cdot (b \cdot \phi)(v). \end{aligned}$$

Theorem 9.1 (Double centralizer theorem)**.** *Let M be a vector space, and $A \subseteq \mathrm{End}(M)$. Then the algebra $B = \mathrm{End}_A(M)$ is semisimple, one has $\mathrm{End}_B(M) = A$, and M has the multiplicity-free complete decomposition*

$$M \cong \bigoplus_{\widehat{M}} A^\lambda \otimes B^\lambda \tag{9.2}$$

as an (A, B)-bimodule, where $\{B^\lambda \mid \lambda \in \widehat{M}\}$ are mutually non-isomorphic irreducible B-modules.

Proof. The proof amounts to showing that $B^\lambda = \mathrm{Hom}(A^\lambda, M)$ is irreducible, and that for $\lambda, \mu \in \widehat{A}$, $\mathrm{Hom}(A^\lambda, M) \cong \mathrm{Hom}(A^\mu, M)$ if and only if $\mu = \lambda$. □

In other words, $(AB)^\lambda = A^\lambda \otimes B^\lambda$ is an irreducible (A, B)-bimodule which is uniquely determined by λ, and which decomposes as

$$\begin{aligned} (AB)^\lambda &= m_M(\lambda) A^\lambda && \text{as an } A\text{-module, and} \\ (AB)^\lambda &= \dim(A^\lambda) B^{M,\lambda} && \text{as a } B\text{-module.} \end{aligned}$$

9.1. **First example: Type A_r and the symmetric group.** There are actually lots of versions if this particular example. In the place of A, we can put any of

$$\begin{array}{ll}
\text{the group algebra } \mathbb{C}\mathrm{GL}_n(\mathbb{C}) & \text{where } \mathrm{GL}_n(\mathbb{C}) \text{ is the group } \{g \in M_n(\mathbb{C}) \mid \det(g) \neq 0\}, \\
\text{the group algebra } \mathbb{C}\mathrm{SL}_n(\mathbb{C}) & \text{where } \mathrm{SL}_n(\mathbb{C}) \text{ is the group } \{g \in M_n(\mathbb{C}) \mid \det(g) = 1\}, \\
\text{the enveloping algebra } U\mathfrak{gl}_n(\mathbb{C}) & \text{where } \mathfrak{gl}_n(\mathbb{C}) \text{ is the Lie algebra } \{x \in M_n(\mathbb{C})\}, \text{ or} \\
\text{the enveloping algebra } U\mathfrak{sl}_n(\mathbb{C}) & \text{where } \mathfrak{sl}_n(\mathbb{C}) \text{ is the Lie algebra } \{x \in M_n(\mathbb{C}) \mid \mathrm{tr}(x) = 0\}.
\end{array}$$

In the place of M, we start with the defining representation $V = \mathbb{C}^n$ of A. We saw in Section 3.1 that both group algebras and enveloping algebras are Hopf algebras, so that

$$\underbrace{V \otimes \cdots \otimes V}_{k \text{ factors}} = V^{\otimes k}$$

is also an A-module (with slightly different actions). Most of the proofs you will run across, for example in [GW, §4] or [FH, §6], will use the case where $A = \mathbb{C}\mathrm{GL}_n$. However, the representation theory of all four algebras is strikingly similar. Since we know the representation theory of $A = U\mathfrak{sl}_n$ the best out of these four examples, we will focus on that example here.

Fix $A = U\mathfrak{sl}_n$. We saw on the homework that $V \cong L(\omega_1) = L(\square)$. In (8.8), we saw that for any $\lambda \in P^+$,

$$L(\lambda) \otimes L(\square) = \sum_{\mu \in \lambda^+} L(\mu)$$

where

$$\lambda^+ = \{ \text{ partitions obtained from } \lambda \text{ by adding a box } \}.$$

Further, tensor products are distributive, namely for some $\lambda_{(1)}, \dots, \lambda_{(\ell)}$,

$$\left(\bigoplus_{i=1}^{\ell} L(\lambda_{(i)}) \right) \otimes L(\square) = \bigoplus_{i=1}^{\ell} (L(\lambda_{(i)}) \otimes L(\square)) = \bigoplus_{\substack{i=1,\dots,\ell \\ \mu \in \lambda_{(i)}^+}} L(\mu).$$

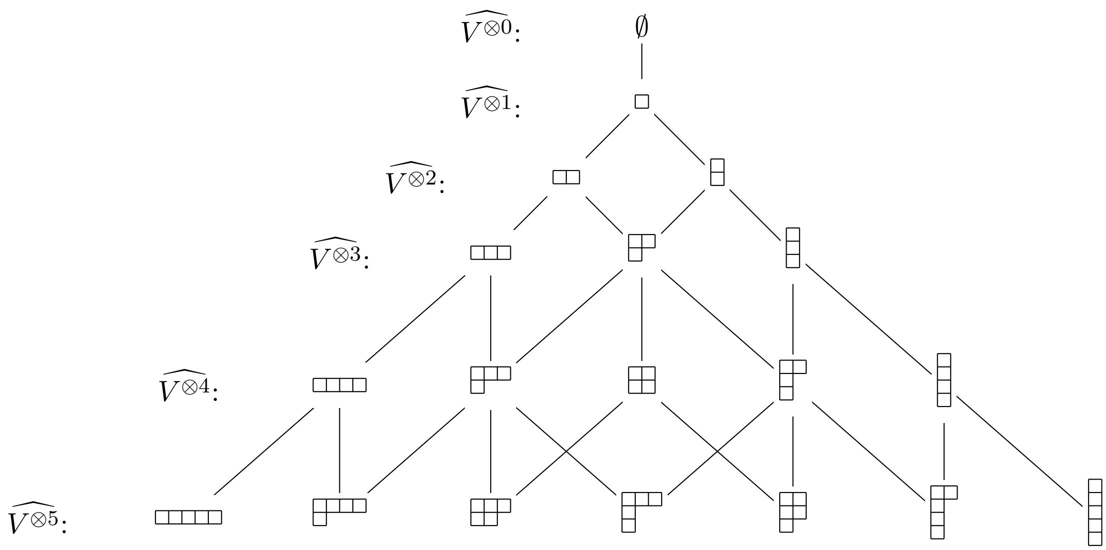
So we can iteratively track the decomposition of $V^{\otimes k}$ by starting with the decomposition of $V^{\otimes(k-1)}$ and and calculating how $L(\lambda) \otimes L(\square)$ contributes for each $\lambda \in \widehat{V^{\otimes(k-1)}}$.

To do this, we build a lattice, called a *Bratteli diagram*, where we put the partitions indexing the isotypic components of $V^{\otimes k}$ on level k, and draw an edge between a partition λ on level $k-1$ and a partition μ on level k if $\mu \in \lambda^+$. Then the multiplicity of the module $L(\mu)$ in $V^{\otimes k}$ is the number of downward moving paths in the lattice, starting at the top. See Figure 4.

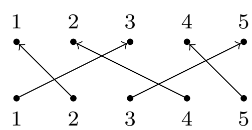
Now take a look at Figure 1, where we saw Young's lattice for the first time. Young's lattice controlled the representation theory of the symmetric group S_k, where paths in the lattice indexed bases for irreducible S_k-modules (rather than multiplicities). Much like we would expect for the centralizer algebra $\mathrm{End}_{U\mathfrak{sl}_n}(V^{\otimes k})$! And yes, in fact, the centralizer we're looking for here is indeed $\mathbb{C}S_k$.

Recall that the symmetric groups S_k is the group of permutations of k objects. We can draw the elements of S_k as diagrams with vertices in a row on top labeled $1, \dots, k$, vertices in a row on bottom labeled $1, \dots, k$, and directed edges from bottom to top forming a bijection. For example,

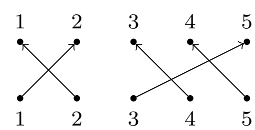
FIGURE 4. Levels 0–5 of the Bratteli diagram for $V^{\otimes k}$ with $V = L(\omega_1)$ and $\mathfrak{g} = \mathfrak{sl}_n$, $n > 5$.



S_5 contains

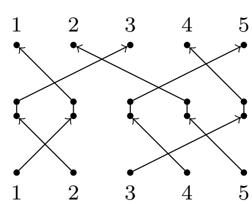
$\sigma_1 =$ 

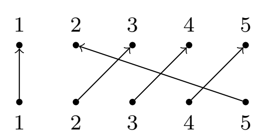
and

$\sigma_2 =$ 

.

Then multiplication is given by concatenation, where $\sigma\tau$ is the result of stacking σ on top of τ and resolving connections. For example,

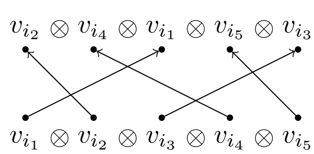
$\sigma_1\sigma_2 =$ 

$=$ 

Then S_k acts on $V^{\otimes k}$ by place permutation. On a simple tensor $v_{i_1} \otimes \cdots \otimes v_{i_k}$, this action is given algebraically by

$$\sigma \cdot (v_{i_1} \otimes \cdots \otimes v_{i_k}) = v_{\sigma^{-1}(i_1)} \otimes \cdots \otimes v_{\sigma^{-1}(i_k)},$$

or diagrammatically by



Recall that the action of $x \in \mathfrak{g}$ on a simple tensor $v_{i_1} \otimes \cdots \otimes v_{i_k}$ looks like

$$x \cdot (v_{i_1} \otimes \cdots \otimes v_{i_k}) = \sum_{j=1}^{k} v_{i_1} \otimes \cdots \otimes (x \cdot v_{i_j}) \otimes \cdots \otimes v_{i_k}.$$

So

$$\begin{aligned} x \cdot (\sigma \cdot (v_{i_1} \otimes \cdots \otimes v_{i_k})) &= \sum_{j=1}^{k} v_{\sigma^{-1}(i_1)} \otimes \cdots \otimes (x \cdot v_{\sigma^{-1}(i_j)}) \otimes \cdots \otimes v_{\sigma^{-1}(i_k)} \\ &= \sigma \sum_{j=1}^{k} v_{\ell_1} \otimes \cdots \otimes (x \cdot v_{\ell_j}) \otimes \cdots \otimes v_{\ell_k} \qquad \text{where } \ell_j = \sigma(i_j) \\ &= \sigma \cdot (x \cdot (v_{i_1} \otimes \cdots \otimes v_{i_k})) \end{aligned}$$

since σ is a bijection. And therefore the image of this action of $\mathbb{C}S_k$ in $\mathrm{End}(V^{\otimes k})$ is contained in $\mathrm{End}_{U\mathfrak{sl}_n}(V^{\otimes k})$.

Example. *Let $A = U\mathfrak{sl}_2$ so that $L(\square) = \mathbb{C}\{v_1, v_2\}$ with $xv_2 = v_1$, $yv_1 = v_2$, and $xv_1 = yv_2 = 0$. Then*

$$L(\square) \otimes L(\square) = \mathbb{C}\{v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_1, v_2 \otimes v_2\}.$$

The weights of this module are $2, 0, 0, -2$ (or $\alpha, 0, 0, -\alpha$) with $(L(\square) \otimes L(\square))_0 = \mathbb{C}\{v_1 \otimes v_2, v_2 \otimes v_1\}$. The module decomposes into $L(\square\square) \oplus L(\substack{\square\\\square}) = L(\square\square) \oplus L(\emptyset)$. So which part of $\mathbb{C}\{v_1 \otimes v_2, v_2 \otimes v_1\}$ is in $L(\square\square)$ and which part is in $L(\substack{\square\\\square})$? Well, let's just look as the action of y on the only weight vector (up to scaling) of weight 2, and that must be the weight vector in $L(\square\square)$ of weight 0:

$$y \cdot (v_1 \otimes v_1) = v_1 \otimes v_2 + v_2 \otimes v_1.$$

The orthogonal complement to $v_1 \otimes v_2 + v_2 \otimes v_1$ in $\mathbb{C}\{v_1 \otimes v_2, v_2 \otimes v_1\}$ is generated by $v_1 \otimes v_2 - v_2 \otimes v_1$, which is indeed annihilated by x and y. So

$$\begin{aligned} L(\square\square) &= \mathbb{C}\{v_1 \otimes v_1, v_1 \otimes v_2 + v_2 \otimes v_1, v_2 \otimes v_2\}, \text{ and} \\ L(\substack{\square\\\square}) &= \mathbb{C}\{v_1 \otimes v_2 - v_2 \otimes v_1\}. \end{aligned}$$

Now let's think about the action of S_2 on this module, where $S_2 = \{1, s \mid s^2 = 1\}$. We have s acting by swapping the factors. Notice that s fixes

$$v_1 \otimes v_1, \quad v_1 \otimes v_2 + v_2 \otimes v_1, \quad \text{and} \quad v_2 \otimes v_2,$$

and so as an S_2-module, $\mathbb{C}\{v_1 \otimes v_1, v_1 \otimes v_2 + v_2 \otimes v_1, v_2 \otimes v_2\}$ is three copies of the trivial module $S^{\square\square}$. On the other hand,

$$s \cdot (v_1 \otimes v_2 - v_2 \otimes v_1) = -(v_1 \otimes v_2 - v_2 \otimes v_1),$$

and so $\mathbb{C}\{v_1 \otimes v_2 - v_2 \otimes v_1\}$ is one copy of the sign representation of S_2, $S^{\substack{\square\\\square}}$. Therefore as a $U\mathfrak{sl}_2, \mathbb{C}S_2$ bimodule,

$$L(\square) \otimes L(\square) = \left(L(\square\square) \otimes S^{\square\square}\right) \oplus \left(L(\substack{\square\\\square}) \otimes S^{\substack{\square\\\square}}\right)$$

To prove that the image of the action of $\mathbb{C}S_k$ in $\mathrm{End}(V^{\otimes k})$ is equal to the whole of $\mathrm{End}_{U\mathfrak{sl}_n}(V^{\otimes k})$, there are two standard approaches. One is via Young symmetrizers and Schur functors, as in [FH, §6]. The other is to to show that the image of the action of $U\mathfrak{sl}_n$ is equal to $\mathrm{End}_{\mathbb{C}S_k}(V^{\otimes k})$, and let Theorem 9.1 to the work. Let $\{v_1, \dots, v_n\}$ be the standard basis of V, so that $\{v_{\mathbf{i}} = v_{i_1} \otimes \cdots \otimes v_{i_k} \mid \mathbf{i} = (i_1, \dots, i_k) \in \{1, \dots, n\}^k\}$ is a basis of $V^{\otimes k}$. Let $b \in \mathrm{End}_{\mathbb{C}S_k}(V^{\otimes k})$ and write its action with respect to this basis as

$$b \cdot v_{\mathbf{i}} = \sum_{\mathbf{j} \in \{1,\dots,n\}^k} b_{\mathbf{i}}^{\mathbf{j}} v_{\mathbf{j}}.$$

Then with $\sigma \cdot \mathbf{i} = (\sigma^{-1}(i_1), \dots, \sigma^{-1}(i_k))$, the relation $\sigma b = b\sigma$ implies

$$\begin{aligned} \sigma \cdot b \cdot v_{\mathbf{i}} &= \sum_{\mathbf{j}} b_{\mathbf{i}}^{\mathbf{j}} v_{\sigma \cdot \mathbf{j}} \\ &= b \cdot \sigma \cdot v_{\mathbf{i}} = \sum_{\mathbf{j}} b_{\sigma\cdot\mathbf{i}}^{\mathbf{j}} v_{\mathbf{j}} = \sum_{\mathbf{j}} b_{\sigma\cdot\mathbf{i}}^{\sigma\cdot\mathbf{j}} v_{\sigma\cdot\mathbf{j}}, \end{aligned}$$

since σ is a bijection on $\{1, \dots, n\}^k$. Comparing coefficients on either side, we get

$$b \in \mathrm{End}_{\mathbb{C}S_k}(V^{\otimes k}) \quad \text{if and only if} \quad b_{\sigma\cdot\mathbf{i}}^{\sigma\cdot\mathbf{j}} = b_{\mathbf{i}}^{\mathbf{j}} \tag{9.3}$$

for all $\sigma \in S_k$ and $\mathbf{i} \in \{1, \dots, n\}^k$. Since the coproduct of $x \in \mathfrak{sl}_n$ is symmetric, this shows that $U\mathfrak{sl}_n \subseteq \mathrm{End}_{\mathbb{C}S_k}(V^{\otimes k})$.

The reverse containment can be done by showing that the trace form on $\mathrm{End}V^{\otimes k}$, restricted to $\mathrm{End}_{\mathbb{C}S_k}(V^{\otimes k})$ is non-degenerate (it's certainly nondegenerate on the image of $U\mathfrak{sl}_n$–we say this early on in this course), and that if $b \in \mathrm{End}_{\mathbb{C}S_k}(V^{\otimes k})$ is orthogonal to $u \in U\mathfrak{sl}_n$, then $b = 0$ (as done in [GW, §4.2.4], for example). However, for an intuitive justification, recall that though $U\mathfrak{sl}_n$ is not isomorphic to $U\mathfrak{gl}_n$, their images in $\mathrm{End}(V)$ are the same; this is because $_n$ has codimension 1 in $\mathfrak{gl}_n$, and is not closed under matrix multiplication. Now $\mathfrak{gl}_n$ is by definition the entire (Lie) algebra of linear maps on $V = \mathbb{C}^n$, i.e. $\mathfrak{gl}_n = \mathrm{End}(V)$. Next, the action of $\mathfrak{gl}_n$ on $V^{\otimes k}$ is exactly the symmetrization of the action go $\mathfrak{gl}_n$ on any factor of $V^{\otimes k}$. So the image of the action of $\mathfrak{gl}_n$ on $V^{\otimes k}$ precisely generates the set of operators $b \in \mathrm{End}(V^{\otimes k})$ which satisfy $b_{\sigma\cdot\mathbf{i}}^{\sigma\cdot\mathbf{j}} = b_{\mathbf{i}}^{\mathbf{j}}$ as in (9.3).

9.2. Back to idempotents. .

The double centralizer theorem says that for a vector space M, $A \subseteq \mathrm{End}(M)$ semisimple, and $B = \mathrm{End}_A(M)$, we have

(1) B is semisimple;
(2) $A = \mathrm{End}_B(M)$; and
(3) as an a, b bimodule,

$$M = \bigoplus_{\lambda \in \widehat{M}} A^\lambda \otimes B^\lambda.$$

Note that

$$Z(A) \subseteq \mathrm{End}_A(B) \qquad \text{and } Z(B) \subseteq \mathrm{End}_B(A).$$

Actually,

$$Z(A) = \mathrm{End}_A(M) \cap A = \mathrm{End}_A(M) \cap \mathrm{End}_B(M) = B \cap \mathrm{End}_B(M) = Z(B).$$

So the center of A is equal to the center of B. In the general case, where A doesn't start off inside of $\mathrm{End}(A)$, and where we find some other algebra B which also acts on M in such a way that these two actions completely centralize each other in $\mathrm{End}(M)$, this says that the image of the action of $Z(A)$ is equal to the image of the action of $Z(B)$.

So in the case where M is finite-dimensional, the proof of the double centralizer decomposition amounts to the fact that the centrally primitive idempotents, which project onto the isotypic components, are the same for both algebras.

In our case, M is usually finite-dimensional, A is a huge algebra (think $U\mathfrak{g}$), but B is a finite-dimensional algebra whose image is often isomorphic to B (think $\mathbb{C}S_k$). So it actually makes sense to go ahead and calculate those centrally primitive idempotents for B.

Computing idempotents. For a good reference on this, see for example [**?**, §7]. Suppose A is a finite-dimensional seimisimple algebra such that the trace form $\langle a, b\rangle = \mathrm{tr}(ab)$ on the regular representation is nondegenerate. Let $\mathcal{B}$ be a basis of A, and $\mathcal{B}^* = \{b^* \mid b \in \mathcal{B}\}$ the dual basis with respect to the trace form. With the irreducible representations of A indexed by $\widehat{A}$, let A^λ be a representative of the class indexed by $\lambda \in \widehat{A}$. Let χ^λ be the map

$$\chi^\lambda : A^\lambda \to \mathbb{C} \qquad \text{defined by} \qquad a \mapsto \mathrm{tr}_{A^\lambda}(a).$$

Then the primitive central idempotent corresponding to A^λ is given by

$$p_\lambda = \frac{1}{c_\lambda} \sum_{b \in \mathcal{B}} \chi^\lambda(b^*) b,$$

where

$$c_\lambda = \frac{1}{\dim(A^\lambda)} \sum_{b \in \mathcal{B}} \chi^\lambda(b) \chi^\lambda(b^*).$$

Appendix A. Bases, roots, and weights for the classical Lie algebras

The general linear Lie algebra is

$$\mathfrak{gl}_r(\mathbb{C}) = \{x \in \mathrm{End}(\mathbb{C}^r)\} \qquad \text{with } [x,y] = xy - yx.$$

The elementary matrices in $\mathfrak{gl}_n$ are denoted by E_{ij} with $E_{ij}v_k = \delta_{jk}v_i$. Let $\langle , \rangle : \mathfrak{h} \otimes \mathfrak{h} \to \mathbb{C}$ be the NIBS form given by

$$\langle x, y\rangle = \mathrm{Tr}(xy) \qquad \text{(using the standard representation).}$$

A.1. **Type A_r.** The Lie algebra of type A_r is

$$\mathfrak{sl}_{r+1}(\mathbb{C}) = \{x \in \mathfrak{gl}_{r+1}(\mathbb{C}) \mid \mathrm{Tr}(x) = 0\}.$$

A triangular basis of $\mathfrak{sl}_{r+1}(\mathbb{C})$ is given by

$$\mathfrak{sl}_{r+1} = \mathfrak{h} \oplus \bigoplus_{1\le i<j\le r+1} \mathfrak{g}_{\alpha_{i,j}} \oplus \bigoplus_{1\le i<j\le r+1} \mathfrak{g}_{-\alpha_{i,j}}$$

with

$$\mathfrak{h} = \mathbb{C}\{h_\ell = E_{\ell,\ell} - E_{\ell+1,\ell+1} \mid 1 \le \ell \le r\}, \quad \mathfrak{g}_{\alpha_{ij}} = \mathbb{C}E_{i,j} \quad \text{and} \quad \mathfrak{g}_{-\alpha_{i,j}} = \mathbb{C}E_{j,i}.$$

Let

$$\varepsilon_i : \mathfrak{h} \to \mathbb{C} \quad \text{be defined by} \quad h \mapsto \mathrm{Tr}(E_{i,i}h)$$

so that $\varepsilon_i(h_\ell) = \delta_{i,\ell} - \delta_{i,\ell+1}$. Then $h_\ell = h_{\varepsilon_\ell - \varepsilon_{\ell+1}}$ and the roots of $\mathfrak{g}$ are given by

$$R = \{\pm\alpha_{i,j} = \pm(\varepsilon_i - \varepsilon_j) \mid 1 \le i < j \le r\}.$$

So

$$\mathfrak{h}^*_{\mathbb{R}} = \{\lambda_1\varepsilon_1 + \cdots + \lambda_{r+1}\varepsilon_{r+1} \mid \lambda_i \in \mathbb{R}, \lambda_1 + \cdots + \lambda_{r+1} = 0\}.$$

A base (associated to the regular weight $\rho = \frac{1}{2}\sum_{i=1}^{r+1}(r+2-2i)\varepsilon_i$) for R is

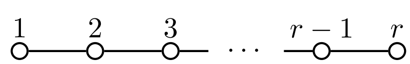
$$B = \{\beta_i = \varepsilon_i - \varepsilon_{i+1} \mid 1 \le i \le r\}, \qquad \text{yielding} \quad R^+ = \{\alpha_{ij} \mid 1 \le i < j \le r+1\}.$$

Since $\langle \beta_i, \beta_i\rangle = 2$, the simple co-roots are $\beta_i^\vee = \beta_i$.

Since

$$\langle \beta_i, \beta_{i\pm1}^\vee\rangle = -1 \qquad \text{and} \qquad \langle \beta_i, \beta_j^\vee\rangle = 0 \text{ for } j \ne i \pm 1,$$

the Coxeter diagram looks like



So the Weyl group is

$$W = \left\langle s_1, \ldots, s_r \mid s_i^2 = 1, s_is_{i+1}s_i = s_{i+1}s_is_{i+1}, s_is_j = s_js_i \text{ for } i \ne \pm j\right\rangle \cong S_{r+1},$$

the symmetric group on $r+1$ letters.

The fundamental weights are

$$\omega_i = \varepsilon_1 + \cdots + \varepsilon_i - \frac{i}{r+1}(\varepsilon_1 + \cdots + \varepsilon_{r+1}), \qquad \text{for } 1 \le i \le r,$$

and the integral weights are

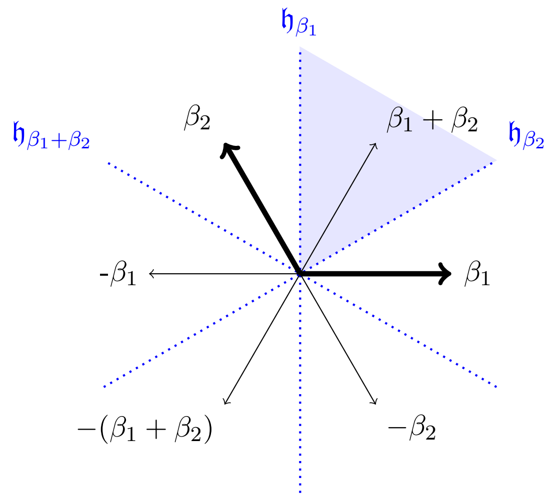
$$P = \{\lambda_1\varepsilon_1 + \cdots + \lambda_r\varepsilon_r - \frac{|\lambda|}{r+1}(\varepsilon_1 + \cdots + \varepsilon_{r+1})\}$$

where

$$\lambda_i \in \mathbb{Z}, \qquad \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_r \ge 0, \qquad \text{and } |\lambda| = \lambda_1 + \cdots + \lambda_r.$$

So the dominant integral weights are in bijection with integer partitions of length less than or equal to r.

The root system for type A_2 looks like



A.2. **Type C_r.** The Lie algebra of type C_r is

$$\mathfrak{sp}_{2r}(\mathbb{C}) = \{x \in \mathfrak{sl}_{2r}(\mathbb{C}) \mid (xu, v) = -(u, xv) \forall u, v \in \mathbb{C}^{2r}\},$$

for a fixed skew symmetric form $(,)L\mathbb{C}^{2r} \otimes \mathbb{C}^{2r} \to \mathbb{C}$. Using $(u, v) = u^T Jv$ with

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

we get the triangular basis

$$\begin{aligned}\mathfrak{sp}_{2r} = \mathfrak{h} \oplus \bigoplus_{1\le i\le r} \mathfrak{g}_{\alpha_i} \oplus \bigoplus_{1\le i<j\le r} (\mathfrak{g}_{\alpha^-_{i,j}} \oplus \mathfrak{g}_{\alpha^+_{i,j}}) \\ \oplus \bigoplus_{1\le i\le r} \mathfrak{g}_{-\alpha_i} \oplus \bigoplus_{1\le i<j\le r} (\mathfrak{g}_{-\alpha^-_{i,j}} \oplus \mathfrak{g}_{-\alpha^+_{i,j}})\end{aligned}$$

with

$$\mathfrak{h} = \{h_i = E_{ii} - E_{r+i,r+i} \mid 1 \le i \le r\},$$
$$\mathfrak{g}_{\alpha_i} = \mathbb{C}E_{i,r+i}, \quad \mathfrak{g}_{\alpha^-_{ij}} = \mathbb{C}(E_{ij} - E_{r+j,r+i}), \quad \mathfrak{g}_{\alpha^+_{ij}} = \mathbb{C}(E_{i,r+j} - E_{j,r+i}),$$
$$\mathfrak{g}_{-\alpha_i} = \mathbb{C}E_{r+i,i}, \quad \mathfrak{g}_{-\alpha^-_{ij}} = \mathbb{C}(E_{ji} - E_{r+i,r+j}), \quad \text{and} \quad \mathfrak{g}_{-\alpha^+_{ij}} = \mathbb{C}(E_{r+j,i} - E_{r+i,j}).$$

Let

$$\begin{aligned}\varepsilon_i : \mathfrak{h} &\to \mathbb{C} && \text{for } i = 1, \ldots, r \\ h_\ell &\mapsto \mathrm{Tr}(E_i i h_\ell) = \delta_{i,\ell}\end{aligned}$$

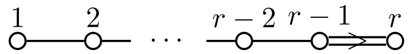
Then

$$R = \{\pm\alpha_k = \pm 2\varepsilon_k, \pm\alpha^-_{ij} = \pm(\varepsilon_i - \varepsilon_j), \pm\alpha^+_{ij} = \pm(\varepsilon_i + \varepsilon_j) \mid 1 \le k \le r, 1 \le i < j \le r\}.$$

Then $h_{\varepsilon_i} = h_i$ and $\mathfrak{h}^*_{\mathbb{R}} = \mathbb{R}^r$. A base for R is

$$B = \{\beta_r = 2\varepsilon_r, \beta_i = \varepsilon_i - \varepsilon_{i+1} \mid 1 \le i \le r-1\}, \qquad \text{yielding} \qquad R^+ = \{\alpha_k, \alpha^{\pm}_{ij} \mid 1 \le k \le r, 1 \le i < j \le r\}.$$

Since $\langle 2\beta_r, 2\beta_r\rangle = 4$ and $\langle \beta_i, \beta_i\rangle = 2$ for $i < r$, the simple co-roots are then $\beta_r^\vee = \frac{1}{2}\beta_r = \varepsilon_r$ and $\beta_i^\vee = \beta_i$ for $i < r$. So the Coxeter diagram looks like



and the Weyl group of type C_r is

$$W = \left\langle s_1, \ldots, s_r \;\middle|\; \begin{array}{c} s_i^2 = 1, s_i s_j = s_j s_i \text{ for } i \neq j \pm 1 \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ for } i \geq 1 \\ s_r s_{r+1} s_r s_{r+1} = s_{r+1} s_r s_{r+1} r \end{array} \right\rangle \cong Z_2 \ltimes S_r,$$

the group of signed permutations on r letters (the subgroup generated by $s_1, \ldots, s_{r-1}$ is the group of permutations; then let s_{r+1} act by flipping the sign of the last element in the permutation). Note that this is reversed from Example 7, but both bases yield isomorphic Weyl groups.

The fundamental weights are

$$\omega_i = \varepsilon_1 + \cdots + \varepsilon_i, \qquad \text{for } 1 \leq i \leq r,$$

and the integral weights are

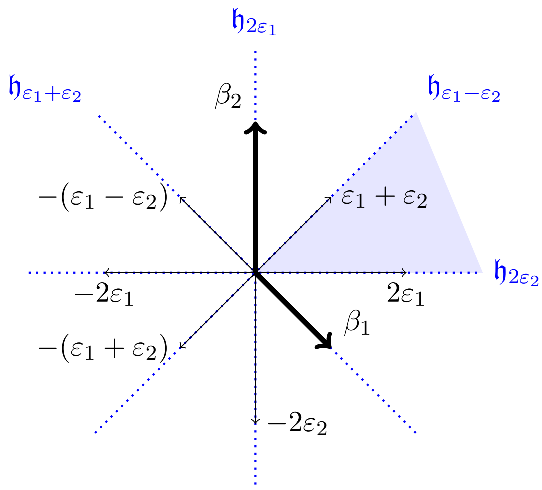
$$P = \lambda_1\varepsilon_1 + \cdots + \lambda_r\varepsilon_r$$

where

$$\lambda_i \in \mathbb{Z}, \qquad \text{and} \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0.$$

So the dominant integral weights are in bijection with integer partitions of length less than or equal to r.

The root system of type C_2 looks like



The difference between this picture and the one for the base in Example 7 is a reflection across the $\mathfrak{h}_{\varepsilon_1-\varepsilon_2}$ hyperplane.

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