Exercise 7: Some things about crystals.

(1) Verify that the two formulas for the Weyl denominator a_ρ agree for type A_2.
(2) Use the three methods from class (Freudenthal's multiplicity formula, the Weyl character formula, and the path model) to calculate the (non-obvious) multiplicities in $L(\omega_2)$ for type C_2. Draw the crystal graph for $\mathcal{B}(\omega_2)$.
(3) In type A_r, we saw that the finite dimensional modules are indexed by integer partitions of length no more than r, that the weight ω_1 corresponds to the integer partition of 1, and that for any $\lambda \in P^+$, the decomposition of $L(\lambda) \otimes L(\omega_1)$ is given by

$$L(\lambda) \otimes L(\omega_1) = \sum_{\mu \in \lambda^+} L(\mu)$$

where λ^+ is the set of all weights corresponding to partitions obtained by adding a box to the partition corresponding to the weight λ.

In type C_r, there's a similar story. Choose the base

$$B = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{r-1} - \varepsilon_r, 2\varepsilon_r\}$$

so that the fundamental weights are given by

$$\omega_i = \varepsilon_1 + \cdots + \varepsilon_i \qquad \text{for } i = 1, \ldots, r.$$

Then

$$P^+ = \left\{ \lambda_1\varepsilon_1 + \cdots + \lambda_r\varepsilon_r \;\middle|\; \begin{array}{c} \lambda_i \in \mathbb{Z} \\ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0 \end{array} \right\}$$

is in bijection with integer partitions of length at most r (with less work than in type A_r, even).

Use the path model to determine the decomposition of $L(\lambda) \otimes L(\omega_1)$ in type C_r, expressing the answer in terms of partitions.